# Layer 2

The central question in layer 2: now that we know what G is, how do we actually extract physics from it?

The answer is: look at where G has poles in the complex plane, and learn to work with $G^{r/a}$ instead of the full time ordered G.

Step 1 — The Lehmann representation: G as a sum over poles (§8 eq. 8.7)
FL9 does something interesting. Starting from the definition of G and inserting a complete set of energy eigenstates, you can write G(ω, p) exactly as:

$$G(\omega,\bar{p}) = \frac{1}{2}(2\pi)^3 \sum_m \left\{ \frac{A_m \delta(\bar{p}-\bar{P}_m)}{\omega + \varepsilon_m(+) + i0} + \frac{B_n \delta(\bar{p}+\bar{P}_n)}{\omega + \varepsilon_n(-) - i0} \right\}$$

Each term is a pole in the complex plane ω. The positions of those poles are $\varepsilon_n(+)$ & $\varepsilon_n(-)$ — the excitation energies of the system when you add or remove a particle respectively. This is the key physical content: the poles of G are the quasiparticles energies.

Notice the ± i0: the particle addition poles sit just below the real axis, and the particle removal poles just above. The sign structure is everything — it is what distinguishes $G^r$ from $G^a$.

## Step 2 - Why the time-ordered G is awkward

The full time-ordered G has poles on both sides of the real axis. This makes it non-analytic which is computationally inconvenient.

$G^{r/a}$ fix this. From §36 eq (36.2) and (36.9)

$$iG^R_{\alpha\beta}(X_1,X_2) = \langle \hat{\psi}_\alpha(X_1)\hat{\psi}^\dagger_\beta(X_2) + \hat{\psi}^\dagger_\beta(X_2)\hat{\psi}_\alpha(X_1)\rangle$$

$$iG^A_{\alpha\beta}(X_1,X_2) = -\langle \hat{\psi}_\alpha(X_1)\hat{\psi}^\dagger_\beta(X_2) + \hat{\psi}^\dagger_\beta(X_2)\hat{\psi}_\alpha(X_1)\rangle$$

The key difference from G is that both $\psi$ and $\psi^\dagger$ appear symmetrically inside the bracket, this is the anticommutator, not the time ordered product. The consequence:

- $G^r(\omega)$ has all its poles strictly in the lower half of the complex plane $\omega$ → it is analytic in the upper half plane.
- $G^a(\omega)$ has all poles strictly in the upper half plane and is analytic in the lower half plane.
- And from eq. (36.10): $G^a(\omega) = [G^r(\omega^*)]^*$

This is why Fransson works with $G^r$ throughout, it is a clean analytic function.

## Step 3 – The spectral function $A(\omega)$

From the Lehmann representation you can show that
$G^r(\omega) - G^a(\omega) = 2i \cdot \mathrm{Im}\, G^r(\omega)$.

This difference defines the spectral function:

$$A(\omega) = i[G^r(\omega) - G^a(\omega)] = -2\,\mathrm{Im}\, G^r(\omega).$$

$A(\omega)$ is real, positive and normalized: $\int A(\omega)\, d\omega/2\pi = 1$. Physically it is the density of states weighted by how strongly the system responds at each frequency. The poles of $G^r$ show up as peaks (or delta functions for sharp states) in $A(\omega)$.

For a free particle with energy $\varepsilon_0$ and a small broadening $\eta$, the retarded Green's function is (eq. 36.17):

$G^r(\omega) = 1/(\omega - \varepsilon_0 + i\eta)$. And the spectral function is:
$A(\omega) = -2\,\mathrm{Im}(G^r(\omega)) = 2\eta/((\omega-\varepsilon_0)^2 + \eta^2)$. This is a Lorentzian centered at $\varepsilon_0$ with width $\eta$. When $\eta \to 0$ it becomes $2\pi\delta(\omega - \varepsilon_0)$, a sharp quasi particle. When $\eta$ is finite, the quasi particle has a finite lifetime $1/\eta$. This $\eta$ is exactly the $\Gamma/2$ in Fransson's single-site Green's function $g^r_m = (\omega - \varepsilon_m + i\Gamma_m/2)^{-1}$

## Step 4 – The relation between $G^{M\wedge}$ and $G^<$

From eq (36.14), at finite temperature in equilibrium:

$$G^r(\omega) = \mathrm{Re}\, G(\omega) + i \coth(\omega/2T) \cdot \mathrm{Im}\, G(\omega).$$

Rearranging and using the fact that $G^<$ is the $t_1 < t_2$ part of $G$, one can show that in equilibrium:

$$G^<(\omega) = i f(\omega) \cdot A(\omega) = i f(\omega) \cdot [-2 \mathrm{Im}\, G^r(\omega)] = 2i f(\omega) \mathrm{Im}\, G^r(\omega)$$

where $f(\omega)$ is the Fermi-Dirac distribution. This is the fluctuation-dissipation theorem for Green's functions. It says: In equilibrium, once you know $G^r$ (the response), you automatically know $G^<$ (the occupation) they are linked by the Fermi function.

This is the last ffq equilibrium result we get. It is important that our system is out of equilibrium - two reservoirs at different chemical potentials. The fluctuation-dissipation theorem breaks down. This is why in layer 4 we pick up the Keldysh formalism.