Layer I – the Green's function as an object.

This is the conceptual bedrock. What a entity is as an operator ~~value~~ expectation value. What $G^{<}$ means physically (occupied vs unoccupied states). This is mostly FFQ and SPT.

The starting point: What problem are we solving?

Perturbation theory term by term is useless for strongly interacting systems. You need to sum infinitely many terms at once. The Green's function is the object that lets you do that. It encodes the full interacting physics in one function.

Step 1 – the field operators (ch 2.2)

Before G is even defined, FFQ introduces the Heisenberg field operators: $\hat{\Psi}_\alpha(t, \vec{r}) = e^{i\hat{H}t}\hat{\psi}_\alpha(\vec{r})e^{-i\hat{H}t}$.

Ψ is a quantum field – it annihilates a spin-α particle at position r and time t. $\Psi^\dagger$ creates one. These are fundamental objects. The index α is the spin label $\uparrow$ or $\downarrow$. In Fransson's notation, he packages them into a spinor $\psi_m = (c_{m\uparrow}, c_{m\downarrow})^T$ at site m, same idea just discretized into a lattice

The key anticommutation rules (eq 7.3) are:

$$\hat{\Psi}_\alpha(t,\bar{r})\hat{\Psi}^\dagger_\beta(t,\bar{r}') + \hat{\Psi}^\dagger_\beta(t,\bar{r}')\hat{\Psi}_\alpha(t,\bar{r}) = \delta_{\alpha\beta}\delta(\bar{r}-\bar{r}')$$

. This is the Fermi statistics, hardwired in. It means two fermions cannot occupy the same state.

Step 2 – The definition of G (eq 7.9)

Now the central definition:

$$G_{\alpha\beta}(\mathbb{X}_1,\mathbb{X}_2) = -i\langle T\,\hat{\Psi}_\alpha(\mathbb{X}_1)\hat{\Psi}^\dagger_\beta(\mathbb{X}_2)\rangle$$

Three things to unpack:

- The angle brackets $\langle \ldots \rangle$ mean expectation value in the ground state (at $T=0$) or the Gibbs state (at finite T).
- T is the time-ordering operator – it automatically puts the later time on the left
- The $-i$ out front is a convention inherited from QFT.

Expanding T explicitly gives eq (7.10):

$$G_{\alpha\beta}(\mathbb{X}_1,\mathbb{X}_2) = \begin{cases} -i\langle \hat{\Psi}_\alpha(\mathbb{X}_1)\hat{\Psi}^\dagger_\beta(\mathbb{X}_2)\rangle & \text{for } t_1 > t_2 \\ i\langle \hat{\Psi}^\dagger_\beta(\mathbb{X}_2)\hat{\Psi}_\alpha(\mathbb{X}_1)\rangle & \text{for } t_1 < t_2 \end{cases}$$

The $t_1 < t_2$ branch is $G^<$ – the lesser Green's function. It equals $+i\langle \hat{\Psi}^\dagger(\mathbb{X}_2)\hat{\Psi}(\mathbb{X}_1)\rangle$. This is exactly a particle density: one particle is created at $\mathbb{X}_2$, propagates, and is destroyed at $\mathbb{X}_1$.

The $t_1 > t_2$ branch is $G^>$ — the greater Green's function. It equals $-i\langle \hat{\psi}(\mathbb{X}_1)\,\hat{\psi}^\dagger(\mathbb{X}_2)\rangle$. A particle is created at $\mathbb{X}_2$ and destroyed at $\mathbb{X}_1$, but the ordering means $\mathbb{X}_1$ is later — this describes the propagation of a hole (an unoccupied state).

Step 3 — the spin structure (eq. 7.11)

This is the equation that makes Fransson's spin-proportional-to-I argument rigorous. It states:

If the system has no preferred spin direction (no magnetization, no external field, no spin-orbit), then:

$$G_{\alpha\beta}(\mathbb{X}_1, \mathbb{X}_2) = \delta_{\alpha\beta}\, G(\mathbb{X}_1, \mathbb{X}_2)$$

($\delta_{\alpha\beta}$: unit matrix; G: scalar function)

The spin matrix structure of G collapses to a unit matrix times a scalar function. This is because there is nothing in the Hamiltonian that can rotate spin — so all spin channels are equivalent, and off-diagonal terms ($G_{\uparrow\downarrow}$) must vanish.*

For instance, in T.A., since H_{mol} has $\bar{V}_m^{(1)} = 0$ in the linear chain case and the reservoirs are non-magnetic, this equation applies at every step.

* Remember, $G_{\uparrow\downarrow}$ would mean that $\downarrow$ electron is created at $\mathbb{X}_2$, propagates through the system and somehow comes out as $\uparrow$. That requires something in the Hamiltonian to have flipped the spin along the way.

Step 4 – G encodes all one-particle physics (eq. 7.15–7.17)

The density matrix is defined as:

$$\rho_{\alpha\beta}(\bar{r}_1, \bar{r}_2) = \frac{1}{N} \langle \hat{\psi}^\dagger_\beta(t, \bar{r}_2)\, \hat{\psi}_\alpha(t, \bar{r}_1) \rangle$$

And from eq. 7.17, comparing with the $t_1 < t_2$ branch of G:

$$\rho_{\alpha\beta}(\bar{r}_1, \bar{r}_2) = -\frac{i}{N} G_{\alpha\beta}(t_1, \bar{r}_1; t_1 + 0, \bar{r}_2)$$

The "0^+" means we approach t_1 from above, this selects the $G^<$-branch. So the density matrix is directly the lesser Green's function at equal times. Any one-particle expectation value – charge density, spin density, current – follows from this.

In Fransson's formula $\langle s_m \rangle = -i \int \mathrm{Tr}(\sigma\, G^<_{mm}(\omega))\, d\omega/4\pi$, the integral over ω is just the Fourier transform at the equal time limit, and the trace over spin with σ is exactly taking the spin component of the density matrix.

Summary of layer I

- G is defined via eq. 7.9, and its two-time ordered branches $G^{(<)}$
- $G^< = e^{i\langle \varphi^\dagger \varphi \rangle}$ is literally the density matrix
- The spin structure collapses to $\delta_{\alpha\beta} \times$ scalar whenever the Hamiltonian has no spin-breaking terms (eq. 2.11)
- Every observable $\langle n \rangle$, $\langle s \rangle$, current, comes from taking a trace of $G^<$ at equal times.