Block IV

What is Ψ_m: the spinor?

In the wave paper $\Psi_m = (\Psi_{m\uparrow}, \Psi_{m\downarrow})^+$ is called a spinor. When the spin is zero, the wave function has one component and is a scalar – invariant under rotations. When the spin is 1/2, the wave function has two components ψ^1 and ψ^2 (correspondig to $\uparrow$ and $\downarrow$). This two-component quantity is what §13 §56 defines as a spinor. Under any rotation of the coordinate system, its components transform linearly:

$$\psi^{1'} = a\psi^1 + b\psi^2, \quad \psi^{2'} = c\psi^1 + d\psi^2$$

With the transformation matrix C satisfying $\det C = 1$ and $C^+ = C^{-1}$, see eq. 56.5 and 56.7.
The three independent parameters in C correspond exactly to the three angles of rotation – so the spinor transformation is fundamentally tied to the geometry of 3D-space.

The critical property, established in §13 §58 eq. (58.1-58.2), is that a rotation through angle φ about an axis n acts on the spinor as:

$$U_n = \exp(i\varphi n \cdot \sigma/2) = \cos(\varphi/2)\cdot I\, e^{i(n\cdot\sigma)}\sin(\varphi/2)$$

Two consequences follow immediately. First, the Pauli matrix vector $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ appears here not as an ad hoc object but as the generator of rotations in spin space. Second, and famously from §58: a rotation through 2π sends $U_u \to -I$, so the spinor components change sign. A spinor requires a full 4π rotation to return to itself, unlike any vector or tensor. This is the mathematical signature of spin-½.

In the model, ψ_m is a spinor at site m. The full molecular spinor $\Psi = (\psi_1, \psi_2, \ldots, \psi_M)^T$ stacks all sites. Everything that follows, the hopping terms, the spin-orbit coupling, the observable $\langle s_m \rangle$ all operates on these spinors.