The signal Hamiltonian $\hat{H}_v(t)$

This represents the coupling between the sensor and the external signal $v(t)$ that we want to measure.

Idea is: signal $v(t)$ $\longmapsto$ perturbation $\hat{H}_v(t)$

$\longmapsto$ change in qubit

$\hat{H}_v(t)$ is usually small. Its presence in the total Hamiltonian perturbs the internal Hamiltonian $\hat{H}_0$, causing the system to deviate from its equilibrium behavior.

The signal Hamiltonian is divided into two components:

$$\hat{H}_v(t) = \hat{H}_{v\parallel}(t) + \hat{H}_{v\perp}(t)$$

where $\parallel$ and $\perp$ mean parallel & perpendicular relative to the qubit's quantization axis (which usually is defined by $\hat{H}_0$).

The parallel component is

$$\hat{H}_{V\parallel}(t) = \frac{1}{2}\beta V_{\parallel}(t)\{|1\rangle\langle 1| - |0\rangle\langle 0|\}$$

It is parallel to the qubit's internal Hamiltonian, so it commutes with $[\hat{H}_{V\parallel}(t), \hat{H}_0] = 0$.

This means that it does not cause transitions between $|0\rangle$ and $|1\rangle$. Instead it changes the relative energy splitting between the two levels. In Pauli notation, since $|1\rangle\langle 1| - |0\rangle\langle 0|$ is a $\hat{\sigma}_z$-type operator, $\hat{H}_{V\parallel}(t)$ can be written as $\hat{H}_{V\parallel}(t) \propto \frac{1}{2}\beta V_{\parallel}(t)\hat{\sigma}_z$, i.e., the term shifts the qubit frequency.

The transverse component is

$$\hat{H}_{V\perp}(t) = \frac{1}{2}\beta\{V_{\perp}(t)|1\rangle\langle 0| + V_{\perp}^{+}(t)|0\rangle\langle 1|\}$$

Unlike the parallel component, this couples two states directly $|0\rangle \leftrightarrow |1\rangle$. $\hat{H}_{V\perp}(t)$ can drive transitions between the qubit levels.

In matrix form, it is an off diagonal perturbation:

$$\hat{H}_{V\perp}(t) = \frac{\beta}{2}\begin{bmatrix} 0 & V_\perp^{+}(t) \\ V_\perp(t) & 0 \end{bmatrix}$$

The full signal Hamiltonian in matrix form is

$$\hat{H}_V(t) = \frac{\beta}{2}\begin{bmatrix} -V_{\parallel}(t) & V_\perp^{+}(t) \\ V_\perp(t) & V_{\parallel}(t) \end{bmatrix}$$

β is a coupling parameter that tells us how strongly the signal couples to the qubit.

Examples are β = Zeeman shift with units Hz/T for magnetic field sensing. Or a linear Stark shift parameter with units Hz/(V/m) for electric field sensing. β converts the physical signal amplitude into a qubit frequency or energy shift. $V_{\parallel}(t)$ changes the energy splitting of the qubit while $V_\perp$ causes transitions/mixing between $|0\rangle$ and $|1\rangle$.