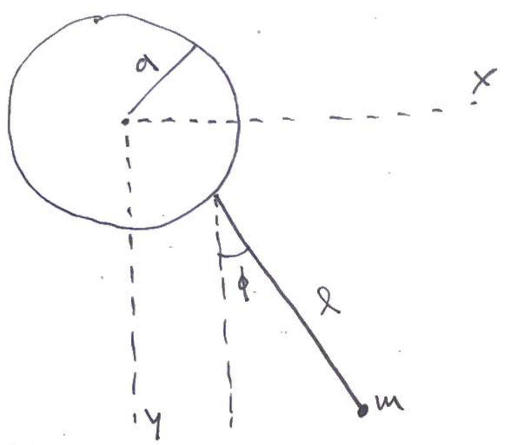
Problem 3.

"A simple pendulum of mass m whose point of support ...."



(a)

Let the mass be located at (x, y) where

$$x = a\cos\gamma t + l\sin\phi \qquad \leadsto \qquad \dot{x} = -a\gamma\sin(\gamma t) + l\dot{\phi}\cos\phi$$
$$y = -a\sin\gamma t + l\cos\phi \qquad\qquad \dot{y} = -a\gamma\cos\gamma t - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = a^2\gamma^2\sin^2\gamma t - 2a\gamma l\dot{\phi}\sin(\gamma t)\cos\phi + l^2\dot{\phi}^2\cos^2\phi$$
$$\dot{y}^2 = a^2\gamma^2\cos^2\gamma t + 2a\gamma l\dot{\phi}\cos\gamma t\sin\phi + l^2\dot{\phi}^2\sin^2\phi$$

using $\sin(\phi - \gamma t) = \sin\phi\cos\gamma t - \cos\phi\sin\gamma t$:

$$T = \tfrac{1}{2}m\left[a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin(\phi - \gamma t)\right] \quad \text{and}$$

$$U = mg(a\sin\gamma t - l\cos\phi) \quad \text{which results in}$$

$$\mathcal{L} = \tfrac{1}{2}m\left[a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin(\phi - \gamma t)\right] - mga\sin\gamma t + mgl\cos\phi$$

Now, since the Lagrangian is defined only to within an additive total time derivative, we can drop the terms only depending on time.

The terms are $\frac{1}{2}ma^2\gamma^2 - mga\sin\gamma t$, which can be dropped. Our Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\sin(\phi-\gamma t) + mgl\cos\phi \quad .$$

However, note that we have a term that mixes $\dot{\phi}$ with explicit time dependence, and this will create messy terms once we compute the EL-equations. To get rid of the mixed term $mal\gamma\dot{\phi}\sin\phi-\gamma t$ we will look for a total time derivative that contains this term but with opposite sign.

E.g., take $\frac{d}{dt}\left[mal\gamma\cos(\phi-\gamma t)\right] = mal\gamma\left[-\sin(\phi-\gamma t)(\dot{\phi}-\gamma)\right]$

$= -mal\gamma\sin(\phi-\gamma t)\dot{\phi} + mal\gamma^2\sin(\phi-\gamma t)$. Adding this to the Lagrangian $L' = L + \frac{d}{dt}\left[mal\gamma\cos(\phi-\gamma t)\right]$

$$\Rightarrow \boxed{L' = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\sin(\phi-\gamma t) + mgl\cos\phi}$$

(b) The point of support now oscillates horizontally with $x = a\cos\gamma t$, hence the coordinates are:

$x = a\cos\gamma t + l\sin\phi$ and $y = l\cos\phi$ with

$\dot{x} = -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi$ and $\dot{y} = -l\dot{\phi}\sin\phi$ which leads to

$\dot{x}^2 + \dot{y}^2 = a^2\gamma^2\sin^2\gamma t - 2al\gamma\dot{\phi}\sin\gamma t\cos\phi + l^2\dot{\phi}^2$

(b) The Lagrangian is then:

$$\mathcal{L} = \frac{1}{2} m a^2 \gamma^2 \sin^2 \gamma t + \frac{1}{2} m l^2 \dot{\phi}^2 - mal\gamma\dot{\phi} \sin\gamma t \cos\phi + mgl\cos\phi$$

Same principle applies, we drop the $\frac{1}{2} m a^2 \gamma^2 \sin^2 \gamma t$ term and use $\frac{d}{dt}\left[-mal\gamma \sin\gamma t \sin\phi\right]$

$$= -mal\gamma^2 \cos\gamma t \sin\phi - mal\gamma\dot{\phi} \sin\gamma t \cos\phi$$ hence.

$$\boxed{\mathcal{L} = \frac{1}{2} m l^2 \dot{\phi}^2 + mal\gamma^2 \cos\gamma t \sin\phi + mgl\cos\phi}$$

(c) Now we have $x = l\sin\phi$ and $y = a\cos\gamma t + l\cos\phi$

s.t. $\dot{x}^2 + \dot{y}^2 = a^2\gamma^2 \sin^2\gamma t + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin\gamma t \sin\phi$ so

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m a^2 \gamma^2 \sin^2\gamma t + mal\gamma\dot{\phi} \sin\gamma t \sin\phi + mga\cos\gamma t + mgl\cos\phi$$

where we can use $\frac{d}{dt}\left[mla\gamma \sin\gamma t \cos\phi\right] = mla\gamma^2 \cos\gamma t \cos\phi + \dots$
$- mla\gamma\dot{\phi}\sin\gamma t \sin\phi$, and in style the unused terms cancel:

$$\boxed{\mathcal{L} = \frac{1}{2} m l^2 \dot{\phi}^2 + mal\gamma^2 \cos\gamma t \cos\phi + mgl\cos\phi.}$$