Problem

"Find the law of transformation of the action S from one inertial frame to another."

The action is defined $S = \int dt\, \mathcal{L}$ for an inertial frame K. Suppose we have another frame K' that moves with ~~...~~ velocity v relative to K. In frame K the velocity of each particle is $\bar{v}_a$ and in K' it is $\bar{v}'_a = \bar{v}_a - \bar{V}$. The kinetic energy T in K is

$$T = \frac{1}{2} \sum m_a v_a^2 = \frac{1}{2} \sum m_a [\bar{v}'_a + \bar{V}]$$

$$= \frac{1}{2} \sum m_a v'^2_a + \bar{V} \cdot \sum m_a \bar{v}'_a + \frac{1}{2} \sum m_a V^2$$

Let $\mu := \sum m_a$, and since U only depends on the relative distances between particles $|\bar{r}_a - \bar{r}_b|$ we get $\bar{r}_a - \bar{r}_b = \bar{r}'_a - \bar{r}'_b$ hence U is invariant under this transformation. The Lagrangian is therefore

$$\mathcal{L} = \mathcal{L}' + \bar{V} \cdot \bar{P}' + \frac{1}{2} \mu V^2 \quad \left(\text{where } \bar{P}' = \sum m_a \bar{v}'_a\right)$$

The action becomes $S = \int dt\, \mathcal{L} = \int dt \left[\mathcal{L}' + \bar{V} \cdot \bar{P}' + \frac{1}{2} \mu V^2\right]$

$= S' + \int \bar{V} \cdot \bar{P}'\, dt + \frac{1}{2} \mu V^2 t$ (where $\int \bar{V} \cdot \bar{P}'\, dt = \int \bar{V} \cdot \sum m_a \bar{v}'_a\, dt$

$= V \cdot \sum m_a r'_a = \mu \bar{V} \cdot \bar{R}'$.) with final result of:

$$\boxed{S = S' + \mu \bar{V} \cdot \bar{R}' + \frac{1}{2} \mu V^2 t}$$