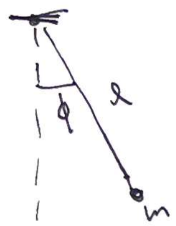
"Determine the period of oscillations of a simple pendulum as a function of the amplitude of oscillations."



The energy of the pendulum is $E = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$. Since we are looking for the period of oscillations we need to find the turning points of the pendulum.

A turning point is when the kinetic energy is zero. Let $\phi = \phi_0$ be such a turning point which in turn gives us the energy $E = -mgl\cos\phi_0$ at these points. Now, going back to the general energy eq. we can solve for $\dot{\phi}$:

$$\dot{\phi} = \sqrt{2[E + mgl\cos\phi]/l^2} \quad \text{where } E = -mgl\cos\phi_0:$$

$$\frac{d\phi}{dt} = \sqrt{\frac{2g}{l}[\cos\phi - \cos\phi_0]} \Rightarrow t = \int \frac{d\phi}{\sqrt{\frac{2g}{l}[\cos\phi - \cos\phi_0]}}.$$

The period T is the time it passes from $-\phi_0$ to ϕ_0 and back, which is equivalent to 4 times the time from 0 to ϕ_0:

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} = \begin{bmatrix} \cos\phi = 1 - 2\sin^2\frac{\phi}{2} \\ \cos\phi_0 = 1 - 2\sin^2\frac{\phi_0}{2} \end{bmatrix}$$

$$= 4\sqrt{\frac{l}{2g}} \cdot \frac{1}{\sqrt{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}} = 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}$$

We want to turn this integral into an elliptical one on the form:

$$K(k) = \int \frac{d\zeta}{\sqrt{1-k^2 \sin^2 \zeta}}$$

starting from $\displaystyle\int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi}{2}}}$ we can substitute

$\sin \frac{\phi}{2} = \sin \frac{\phi_0}{2} \sin \zeta$. The denominator becomes:

$$\sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi}{2} = \sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi_0}{2} \sin^2 \zeta = \sin^2 \frac{\phi_0}{2} (1 - \sin^2 \zeta).$$

Also the differential $d\phi$ in terms of $d\zeta$ is obtained by:

$$d\phi \frac{1}{2} \cos \frac{\phi}{2} = \sin \frac{\phi_0}{2} \cos \zeta \, d\zeta \Rightarrow d\phi = \frac{2 \sin \frac{\phi_0}{2} \cos \zeta}{\cos \frac{\phi}{2}} d\zeta$$

where $\cos \frac{\phi}{2} = \sqrt{1 - \sin^2 \frac{\phi}{2}} = \sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \zeta}$. The integral then becomes:

$$\int_0^{\pi/2} \frac{2 \sin \frac{\phi_0}{2} \cos \zeta}{\sqrt{\sin^2 \frac{\phi_0}{2} (1 - \sin^2 \zeta)}} \cdot \frac{d\zeta}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \zeta}}$$

$$= 2 \int_0^{\pi/2} \frac{d\zeta}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \zeta}} = 2K\left(\sin \frac{\phi_0}{2}\right).$$ The full period

is then $T = 2\sqrt{\frac{l}{g}} \cdot 2K\left(\sin \frac{\phi_0}{2}\right) \boxed{= 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\phi_0}{2}\right)}$. Now,

we might want to look at what happens to T when $\sin \frac{\phi_0}{2}$ is small.

The binomial expansion for small x is $(1+x)^n \approx 1 + nx + \ldots$ which in our case is $(1-k^2\sin^2 z)^{-1/2}$ with $x = -k^2\sin^2 z$ and $n = -\frac{1}{2}$ giving:

$$(1-k^2\sin^2 z)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-k^2\sin^2 z) = 1 + \frac{1}{2}k^2\sin^2 z$$

Then we can integrate the expression for $K(k)$:

$$K(k) \approx \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2\sin^2 z\right) dz = \frac{\pi}{2} + \frac{k^2\pi}{8}$$

$= \frac{\pi}{2}\left(1 + \frac{k^2}{4}\right)$. We have $k = \sin\frac{\phi_0}{2}$ and for small ϕ_0: $k^2 = \sin^2\frac{\phi_0}{2} \approx \frac{\phi_0^2}{4}$ which substituted into K gives $K(\sin\frac{\phi_0}{2}) \approx \frac{\pi}{2}\left(1 + \frac{\phi_0^2}{16}\right)$ and the period T:

$$T = 4\sqrt{\frac{l}{g}}\,K\left(\sin\frac{\phi_0}{2}\right) = 4\sqrt{\frac{l}{g}}\,\frac{\pi}{2}\left(1 + \frac{\phi_0^2}{16}\right)$$

$$\boxed{= 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\phi_0^2}{16}\right)}$$

where ϕ_0^2 is the first correction term to the simple pendulum.