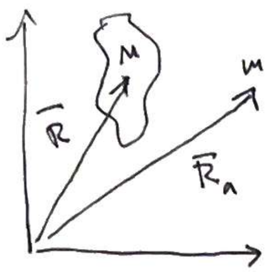
PROBLEM

" A system consists of one particle of mass M and n particles with equal masses m. Eliminate the motion ... "



Denote $\bar{R}$ as the vector to M and $\bar{R}_a$ to the particles with mass m where $a = (1, 2, 3, \ldots n)$.
Let $\bar{r}_a = \bar{R}_a - \bar{R}$ be the relative distance between the bodies and select the origin to be at the centre of mass, i.e.,
$M\bar{R} + m\sum \bar{R}_a = 0$. We want to express $\bar{R}$ and $\bar{R}_a$ in terms of relative vector and CoM:

$M\bar{R} + m\sum(\bar{r}_a + \bar{R}) = 0 \iff \bar{R}(M + mn) = -m\sum\bar{r}_a$ where we let $\mu = M + nm \Rightarrow \bar{R} = -\frac{m}{\mu}\sum\bar{r}_a$ and $\bar{R}_a = \bar{R} + \bar{r}_a$.

The Lagrangian is $\mathcal{L} = \frac{1}{2}M\dot{\bar{R}}^2 + \frac{1}{2}m\sum\dot{\bar{R}}_a^2 - U$

$$\Rightarrow \mathcal{L} = \frac{1}{2}M\dot{\bar{R}}^2 + \frac{1}{2}m\sum[\dot{\bar{R}} + \dot{\bar{r}}_a]^2 - U = \frac{1}{2}M\dot{\bar{R}}^2 + \frac{1}{2}m\sum\dot{\bar{R}}^2 + m\sum\dot{\bar{R}}\dot{\bar{r}}_a + \frac{1}{2}m\sum\dot{\bar{r}}_a^2 - U$$

$= \frac{1}{2}M\dot{\bar{R}}^2 + \frac{1}{2}m\sum\dot{\bar{R}}^2 + m\dot{\bar{R}}\sum\dot{\bar{r}}_a + \frac{1}{2}m\sum\dot{\bar{r}}_a^2 - U$. To get rid of the mixed term with $\dot{\bar{R}}$ and $\dot{\bar{r}}_a$ we can differentiate the COM condition:

$M\dot{\bar{R}} + m\sum\dot{\bar{R}}_a = 0 \Rightarrow M\dot{\bar{R}} + m\sum(\dot{\bar{r}}_a + \dot{\bar{R}}_a) = 0 \Rightarrow m\sum\dot{\bar{r}}_a = -\mu\dot{\bar{R}}$

our Lagrangian becomes $\mathcal{L} = \frac{1}{2}M\dot{\bar{R}}^2 + \frac{1}{2}m\sum\dot{\bar{R}}^2 - \mu\dot{\bar{R}}^2 + \frac{1}{2}m\sum\dot{\bar{r}}_a^2 - U$

$= \frac{1}{2}\dot{\bar{R}}^2[M + nm - 2\mu] + \frac{1}{2}m\sum\dot{\bar{r}}_a^2 - U = -\frac{1}{2}\frac{m^2}{\mu}(\sum\dot{\bar{r}}_a)^2 + \frac{1}{2}m\sum\dot{\bar{r}}_a^2 - U$

Landau sets $\bar{v}_a := \dot{\bar{r}}_a$ s.t. the final result is:

$$\boxed{\mathcal{L} = \frac{1}{2}m\sum v_a^2 - \frac{1}{2}\frac{m^2}{\mu}(\sum v_a)^2 - U}$$