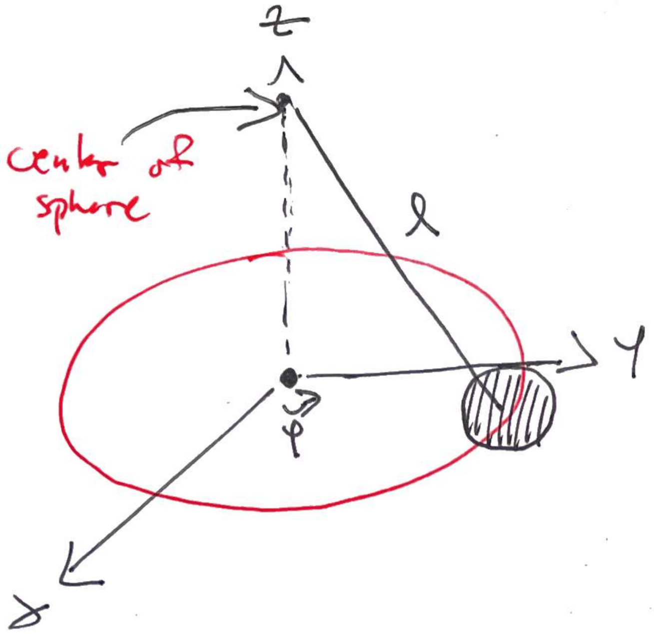
§14, Problem 7



A natural choice of coordinates is evidently spherical ones where we choose z-axis as positive downwards.

From $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ & $z = r\cos\theta$ the kinetic energy is $T = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$. The potential is derived from $\vec{F} = -\partial U/\partial\vec{r}$:

$$F = -\partial U/\partial z \Rightarrow U = -mgz$$

Thus the Lagrangian is $L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta$. Observe that ϕ is a cyclic coordinate, which means that $p_\phi = \text{const}$. The conjugate momentum p_ϕ is given by

$p_\phi = \partial L/\partial\dot{\phi} = ml^2\dot{\phi}\sin^2\theta = \text{const}$. This is useful since we can now express $\dot{\phi}$ in terms of θ: $\dot{\phi} = \dfrac{p_\phi}{ml^2\sin^2\theta}$.

Since the Lagrangian has no explicit time dependence, energy is conserved. The energy is

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2\theta \, \dot{\phi}^2 - mgl\cos\theta = \text{const}$$

Now, replacing $\dot{\phi}$ using conserved momentum:

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

where $U_{eff}(\theta) = \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$. The result of this is that ϕ and $\dot{\phi}$ are gone. We have only one coordinate θ, formally a 1D-problem. Solving for $\dot{\theta} = \frac{d\theta}{dt}$:

$$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

$$\Rightarrow \quad t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}}$$

To get $\phi(t)$, consider $\dot{\phi}$ as a function of θ in the previous page and integrate it. We had

$$d\phi / d\theta = \frac{p_\phi / (ml^2\sin^2\theta)}{\sqrt{\frac{2}{ml^2}(E - U_{eff})}} \quad \text{which gives}$$

$$\phi = \int \frac{p_\phi \, d\theta}{ml^2\sin^2\theta\sqrt{\frac{2}{ml^2}(E - U_{eff})}}.$$

The shape of the path on the sphere.