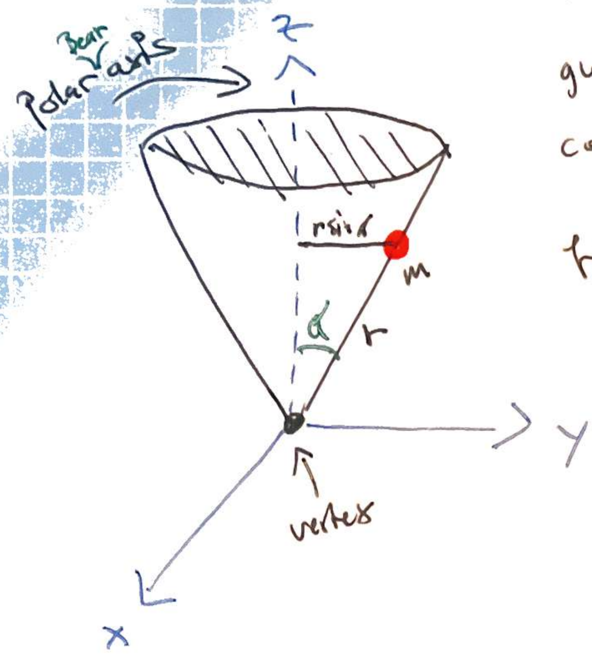
§14. Problem 2



I'm bored out of my mind, I guess this is ripe for spherical coordinates again!

Let $x = r \sin\alpha \cos\varphi$
$y = r \sin\alpha \sin\varphi$
$z = r\cos\alpha$.

Note that α is fixed, so $\dot{\alpha} = 0$, hence the kinetic energy is $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2\sin^2\alpha)$. Taking the polar axis positive upward, $\vec{F} = -mg\hat{z}$. The force is connected to the potential energy by the relation

$\vec{F} = -\frac{\partial U}{\partial \vec{r}}$ s.t. $U = -\int \vec{F}\cdot d\vec{r}$. The upper limit

in the integral can be chosen arbitrarily. If the force tends to zero at ∞ faster than $\frac{1}{r}$ then $\int_r^\infty F\,dr$ is convergent. Thus we put:

$U(r) = \int_r^\infty F\,dr$, $U(\infty) = 0$. The choice of arbitrary constant here is called the gauge. Anywho,

$U(r) = -\int_z^\infty mg\,dz = mgz$.

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\varphi}^2 \sin^2 \alpha) - mgr\cos\alpha$$

Numbing through repetition, the cyclic coordinate is φ, hence $p_\varphi = \text{const}$. This means that

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \sin^2\alpha \Leftrightarrow \dot{\varphi} = \frac{p_\varphi}{m r^2 \sin^2 \alpha}.$$

$\mathcal{L}$ has no explicit t so energy is conserved.

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 \sin^2\alpha + mgr\cos\alpha$$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{p_\varphi^2}{2 m r^2 \sin^2 \alpha} + mgr\cos\alpha}_{= U_{eff}(r)}$$

Back to 1D business, we can solve for t by:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - U_{eff}(r)]}} \quad \text{and}$$

$$\varphi = \frac{p_\varphi}{\sqrt{2m}\sin^2\alpha} \int \frac{dr}{r^2 \sqrt{E - U_{eff}(r)}}$$

Interchange $M_z = p_\varphi$ as you like, nonetheless, the standard answer.