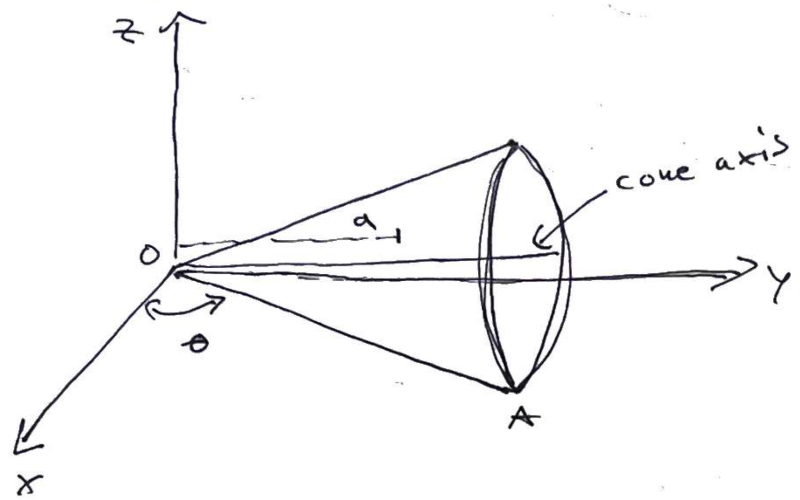
PROBLEM 7.

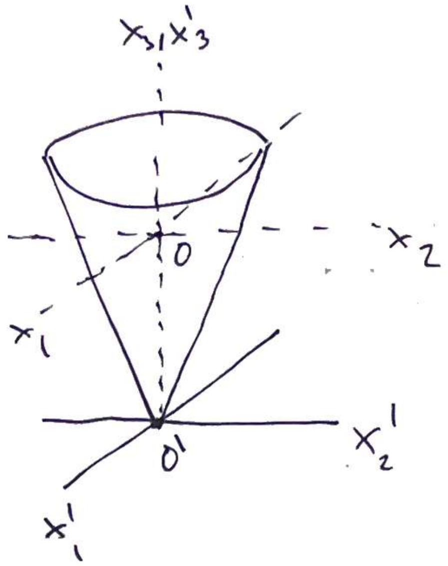
"Find the kinetic energy of a homogeneous cone rolling on a plane."



Generally, $T = \frac{1}{2}\mu V^2 + \frac{1}{2} I \Omega^2$, so our task is to express V, Ω & I properly. To find V we have to understand that the cone axis of symmetry is not in the plane, hence we need to project $a \cdot \dot{\theta} \mapsto a\dot{\theta}\cos\alpha = V$, where α is the angle between the cone axis and plane. The no-slipping condition yields $\Omega = \dfrac{V}{a\sin\alpha} = \dot{\theta}\cot\alpha$. Realise that $\dot{\theta}$ is the turning of the cone in the plane whereas Ω is the rotation about line OA.

The issue of computing I is done in problem 2(e) but let's do it again.

Consider the following:



The idea is to compute the moment of inertia at the vertex and then translate it to COM via Steiner's theorem.

Some general remarks, cylindrical coordinates give $dV = r\,dr\,d\phi\,dz$ and ~~therefore~~ $\rho = \frac{\mu}{V} = \frac{\mu}{\frac{\pi}{3}R^2 h}$.

The size of the radius changes as we go up vertically which means $r(z) = \frac{R}{h}z$ (where $x_3' = z$). Evidently,

$$I_3' = \int r^2\,dm = \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{R}{h}z} r^2 \cdot r\,dr\,d\phi\,dz$$

$$= \rho \frac{\pi R^2 h}{10} = \frac{\mu}{\frac{1}{3}\pi R^2 h} \frac{\pi R^2 h}{10} = \frac{3}{10}\mu R^2.$$

For I_1', take a point relative ~~on~~ the x_1' axis, the distance to it is $y^2 + z^2$ where $y^2 = r^2 \sin^2\phi$.
(from the "point")

$$I_1' = \rho \int_0^{2\pi} \int_0^h \int_0^{\frac{R}{h}z} (r^2 \sin^2\phi + z^2)\, r\,dr\,d\phi\,dz \text{ , split into A \& B:}$$

$$A = \rho \int_0^{2\pi} \int_0^h \int_0^{\frac{R}{h}z} r^3 \sin^2\phi\,dr\,d\phi\,dz = \rho \frac{R^4}{4h^4} \int_0^{2\pi} \int_0^h z^4 \sin^2\phi\,dz\,d\phi$$

$$= \rho \frac{R^4 h}{20} \int_0^{2\pi} \sin^2\phi\,d\phi = \rho \frac{\pi R^4 h}{20}.$$

And $B = \rho \int_0^{2\pi} \int_0^{h} \int_0^{\frac{R}{h}z} z^2 r \, dr \, d\phi \, dz$

$$= \rho \frac{R^2}{2h^2} \int_0^{2\pi} \int_0^{h} z^4 \, d\phi \, dz = \rho \frac{R^2 h^3}{10} \cdot 2\pi$$

$$= \frac{\rho R^2 h^3 \pi}{5}$$

$$A + B = \frac{\mu}{\frac{1}{3}\pi R^2 \cdot h} \left[\frac{R^4 h}{20} \pi + \frac{R^2 h^3 \pi}{5} \right] = \frac{3}{5} \mu \left[\frac{R^2}{4} + h^2 \right]$$

where $I_2' = I_1'$ due to symmetry. The next step is to translate I_i' to I_i. Clearly $I_3 = I_3'$ due to cone symmetry, but for I_1 & I_2 we need to find the distance to COM on the axis.

By <u>def:</u> $z_{cm} = \frac{1}{\mu} \int z \, dm$ where $dm = \rho \, dV$ $= \rho A \, dz = \rho \pi \frac{R^2}{h^2} z^2 \, dz$.

$$z_{cm} = \frac{1}{\mu} \int_0^h z \, dm = \frac{\rho}{\mu} \int_0^h z \pi \frac{R^2}{h^2} z^2 \, dz = \frac{3h}{4}.$$

The steiner theorem says that $I' = I + \mu d^2$ $\Rightarrow I = I' - \mu d^2$, where d is the perpendicular distance between axes.

A quick calculation shows that:

$$I_1 = I_1' - \mu\left(\frac{3}{4}h\right)^2 = \frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right) - \mu\frac{9}{16}h^2$$

$$= \frac{3R^2\mu}{20} + \frac{3}{80}\mu h^2 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$$

Same applies to I_2, i.e., $I_1 = I_2$. Finally, from the general expression of the kinetic energy:

$T = \frac{1}{2}\mu V^2 + \frac{1}{2}I\Omega^2$ we can identify that

$\frac{1}{2}I\Omega^2 = \frac{1}{2}\left[I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right]$. The vector $\vec{\Omega}$ is along OA which is tilted which means, relative to the cone's principal axes:

$\Omega_3 = \Omega\cos\alpha$ (along the cone axis x_3)

$\Omega_1 = \Omega\sin\alpha$ (perp to the cone axis)

So, we get $T = \frac{1}{2}\mu a^2\dot{\theta}^2\cos^2\alpha + \frac{1}{2}I_1(\Omega\sin\alpha)^2 + \frac{1}{2}I_3(\Omega\cos\alpha)^2$

and since $\Omega = \dot{\theta}\cot\alpha$:

$$T = \frac{1}{2}\mu a^2\dot{\theta}^2\cos^2\alpha + \frac{1}{2}I_1\dot{\theta}^2\cos^2\alpha + \frac{1}{2}I_3\dot{\theta}^2\frac{\cos^4\alpha}{\sin^2\alpha}$$

We need to make further simplification using the values we obtained earlier.

using $a = \frac{3}{4}h$, $I_1 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$, $I_3 = \frac{3}{10}\mu R^2$

$$T = \frac{1}{2}\mu\left(\frac{3h}{4}\right)^2 \dot{\theta}^2 \cos^2\alpha + \frac{1}{2}\cdot\frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)\dot{\theta}^2\cos^2\alpha + \dots$$

$$+ \frac{1}{2}\,\frac{3}{10}\mu R^2\,\dot{\theta}^2\,\frac{\cos^4\alpha}{\sin^2\alpha}$$

$$= \frac{9}{32}\mu h^2\dot{\theta}^2\cos^2\alpha + \frac{3}{40}\mu\left(R^2 + \frac{h^2}{4}\right)\dot{\theta}^2\cos^2\alpha + \frac{3}{20}\mu R^2\dot{\theta}^2\,\frac{\cos^4\alpha}{\sin^2\alpha}$$

Typical for cone geometry is $\tan\alpha = \frac{R}{h} \Rightarrow R = h\tan\alpha$:

$$T = \frac{9}{32}\mu h^2\dot{\theta}^2\cos^2\alpha + \frac{3}{40}\mu\left(h^2\tan^2\alpha + \frac{h^2}{4}\right)\dot{\theta}^2\cos^2\alpha + \dots$$

$$+ \frac{3}{20}\mu h^2\tan^2\alpha\,\dot{\theta}^2\,\frac{\cos^4\alpha}{\sin^2\alpha}$$

$$= \mu h^2\dot{\theta}^2\left[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\left(\tan^2\alpha + \frac{1}{4}\right)\cos^2\alpha + \frac{3}{20}\tan^2\alpha\,\frac{\cos^4\alpha}{\sin^2\alpha}\right]$$

$$= \mu h^2\dot{\theta}^2\left[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\left(\tan^2\alpha + \frac{1}{4}\right)\cos^2\alpha + \frac{3}{20}\cos^2\alpha\right]$$

$$= \mu h^2\dot{\theta}^2\left[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\sin^2\alpha + \frac{3}{160}\cos^2\alpha + \frac{3}{20}\cos^2\alpha\right]$$

$$= \left[\frac{45}{160} + \frac{3}{160} + \frac{24}{160} = \frac{72}{160} = \frac{9}{20}\right]$$

$$= \mu h^2\dot{\theta}^2\left[\frac{9}{20}\cos^2\alpha + \frac{3}{40}\sin^2\alpha\right]$$

$$= \left[\sin^2\alpha = 1 - \cos^2\alpha \right]$$

$$= \mu h^2 \dot{\theta}^2 \left[\frac{9}{20}\cos^2\alpha + \frac{3}{40}(1-\cos^2\alpha) \right]$$

$$= \mu h^2 \dot{\theta}^2 \left[\frac{15}{40}\cos^2\alpha + \frac{3}{40} \right]$$

$$= \frac{3\mu h^2 \dot{\theta}^2}{40}(1 + 5\cos^2\alpha).$$

And I'm happy to say that the final result is

$$\boxed{T = \frac{3\mu h^2 \dot{\theta}^2}{40}\left[1 + 5\cos^2\alpha\right]}$$

$\square$