Problem 4.

"Find the deflection of a free falling body from the vertical caused by the Earth's rotation, assuming the angular velocity of this rotation to be small."

The general Lagrangian for non-inertial frame is:

$$\mathcal{L} = \frac{1}{2}mv^2 + m\bar{v}\cdot\bar{\Omega}\times\bar{r} + \frac{1}{2}m(\bar{\Omega}\times\bar{r})^2 - m\bar{W}\cdot\bar{r} - U.$$

We will compute $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \bar{v}} = \frac{\partial \mathcal{L}}{\partial \bar{r}}$:

$$\frac{d}{dt}\left[m\bar{v} + m\bar{\Omega}\times\bar{r}\right] = m\bar{v}\times\bar{\Omega} + m(\bar{\Omega}\times\bar{r})\times\bar{\Omega} - m\bar{W} - \frac{\partial U}{\partial \bar{r}}$$

$$\Leftrightarrow$$

$$m\dot{\bar{v}} = m\bar{v}\times\bar{\Omega} - m\bar{\Omega}\times\bar{v} + m(\bar{\Omega}\times\bar{r})\times\bar{\Omega} - m\bar{W} - \frac{\partial U}{\partial \bar{r}} - m\dot{\bar{\Omega}}\times\bar{r}$$

$$= -\frac{\partial U}{\partial \bar{r}} - m\bar{W} + m\bar{r}\times\dot{\bar{\Omega}} + 2m\bar{v}\times\bar{\Omega} + m\bar{\Omega}\times(\bar{r}\times\bar{\Omega})$$

For this problem we take $\bar{\Omega} =$ const (uniform rotation) and $\bar{W} = 0$ since there is no translational acceleration. The Lagrangian therefore becomes:

$$\mathcal{L} = \frac{1}{2}mv^2 + m\bar{v}\cdot\bar{\Omega}\times\bar{r} + \frac{1}{2}m(\bar{\Omega}\times\bar{r})^2 - U$$ and hence

$$m\dot{\bar{v}} = -\frac{\partial U}{\partial \bar{r}} + 2m\bar{v}\times\bar{\Omega} + m\bar{\Omega}\times(\bar{r}\times\bar{\Omega}).$$ For a falling body, $U = -m\bar{g}\cdot\bar{r} \Rightarrow -\partial U/\partial \bar{r} = m\bar{g}$ and since $\bar{\Omega}$ is small we get: $\dot{\bar{v}} = 2\bar{v}\times\bar{\Omega} + \bar{g}$. We want to find $\bar{v}$, but it is currently coupled on the RHS. Here, since Ω small, we can treat this as a perturbation problem with small parameter Ω.

Zeroth order $\bar{\Omega} = 0$ yields $\dot{\bar{v}}_1 = \bar{g}$ (free fall, no rotation). Integrating with $v_0 = 0$ $\Rightarrow$ $\bar{v}_1 = \bar{g}t$. Now write $\bar{v} = \bar{v}_1 + \bar{v}_2$ where $\bar{v}_2$ is a small correction due to rotation. substitution into the full equation:

$$\dot{\bar{v}}_1 + \dot{\bar{v}}_2 = 2(\bar{v}_1 + \bar{v}_2) \times \bar{\Omega} + \bar{g}$$ where $\dot{\bar{v}}_1 = \bar{g}$ gives:

$$\dot{\bar{v}}_2 = 2\bar{v}_1 \times \bar{\Omega} + 2\bar{v}_2 \times \bar{\Omega}$$ we are only working with the first order of Ω and $\bar{v}_2 \times \bar{\Omega}$ has order 2 in Ω we drop it!

$$\dot{\bar{v}}_2 = 2\bar{v}_1 \times \bar{\Omega} = 2(\bar{g}t) \times \bar{\Omega} = 2t\,\bar{g} \times \bar{\Omega}$$

Integrating once: $\bar{v}_2 = t^2 \bar{g} \times \bar{\Omega}$, so the full velocity is $\bar{v} = \bar{v}_1 + \bar{v}_2 = \bar{g}t + t^2 \bar{g} \times \bar{\Omega}$. Now assume free fall starts from $r = h$, integrating $\bar{v}$ then gives:

$$\bar{r} = h + \frac{1}{2}\bar{g}t^2 + \frac{1}{3}t^3 \bar{g} \times \bar{\Omega}$$, with the third term being the Coriolis deflection. To compute $\bar{g} \times \bar{\Omega}$ we need a coordinate system, choose z to be vertically upward, x northward along the surface and y ~~east~~ westward.

Gravity $\bar{g} = (0, 0, -g)$ and for $\bar{\Omega}$ consider that earth's rotation axis points toward the North pole but the z-axis points radially outward. So at latitude λ these directions differ by angle λ, i.e., $\bar{\Omega} = (-\Omega\cos\lambda, 0, \Omega\sin\lambda)$. computing

$$\bar{g} \times \bar{\Omega} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & -g \\ \Omega\cos\lambda & 0 & \Omega\sin\lambda \end{vmatrix} = -\hat{y}(g\Omega\cos\lambda) = -g\Omega\cos\lambda\,\hat{y}.$$

The deflection is purely in the y-direction, in other words:

$$y = -\frac{1}{3} t^3 g \Omega \cos\lambda$$

The body is dropped from height h and reaches the ground when $z = 0 \Rightarrow \frac{1}{2} g t^2 = h \Rightarrow t = \sqrt{2h/g}$.
Substituting this into y:

$$\boxed{y = -\frac{1}{3}\left(\frac{2h}{g}\right)^{3/2} g \Omega \cos\lambda}$$

with the negative sign indicating eastward deflection!