Problem 7.

"Find the Hamiltonian for a single particle in Cartesian, cylindrical and spherical co-ordinates."

In cartesian coordinates, $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$.
The generalized momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, so we get $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, $p_y = \partial\mathcal{L}/\partial\dot{y} = m\dot{y}$ and $p_z = \partial\mathcal{L}/\partial\dot{z} = m\dot{z}$.
Invert these: $\dot{x} = \frac{p_x}{m}$, $\dot{y} = \frac{p_y}{m}$ and $\dot{z} = \frac{p_z}{m}$.

The Hamiltonian is given by $H = \sum p_i\dot{q}_i - \mathcal{L}$:

$$H = \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x,y,z)$$

$$\boxed{= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x,y,z)}.$$

For cylindrical coordinates (r, ϕ, z):

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases} \leadsto \begin{cases} \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{z} = \dot{z} \end{cases}$$

where $\dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2\phi - 2\dot{r}r\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\sin\phi + \dot{r}^2\sin^2\phi + \ldots$

$$+ 2\dot{r}r\dot{\phi}\sin\phi\cos\phi + r^2\dot{\phi}^2\cos^2\phi = \dot{r}^2 + r^2\dot{\phi}^2$$

The Lagrangian in cylindrical coordinates is then:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2 \right) - U(r, \phi, z).$$

The momenta are: $p_r = \partial \mathcal{L} / \partial \dot{r} = m\dot{r}$, $p_\phi = \partial \mathcal{L} / \partial \dot{\phi} = m r^2 \dot{\phi}$ and $p_z = \partial \mathcal{L} / \partial \dot{z} = m\dot{z}$.

Inverting: $\dot{r} = \frac{p_r}{m}$, $\dot{\phi} = \frac{p_\phi}{m r^2}$, $\dot{z} = \frac{p_z}{m}$. The Hamiltonian becomes:

$$H = \frac{p_r^2}{m} + \frac{p_\phi^2}{m r^2} + \frac{p_z^2}{m} - \frac{p_r^2}{2m} - \frac{p_\phi^2}{2 m r^2} - \frac{p_z^2}{2m} + U(r, \phi, z)$$

$$\boxed{= \frac{1}{2m} \left(p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2 \right) + U(r, \phi, z)}$$

In spherical coordinates (r, θ, ϕ):

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \rightsquigarrow \begin{cases} \dot{x} = \dot{r} \sin\theta \cos\phi + r \dot{\theta} \cos\theta \cos\phi - r\dot{\phi} \sin\theta \sin\phi \\ \dot{y} = \dot{r} \sin\theta \sin\phi + r\dot{\theta} \cos\theta \sin\phi + r \dot{\phi} \sin\theta \cos\phi \\ \dot{z} = \dot{r} \cos\theta - r\dot{\theta} \sin\theta \end{cases}$$

$$\dot{x}^2 = \dot{r}^2 \sin^2\theta \cos^2\phi + r^2 \dot{\phi}^2 \cos^2\theta \cos^2\phi + r^2 \dot{\phi}^2 \sin^2\phi \sin^2\phi$$
$$+ 2 r \dot{r} \sin\theta \cos\theta \cos^2\phi - 2 r \dot{r} \dot{\phi} \sin^2\theta \cos\phi \sin\phi + \ldots$$
$$- 2 r^2 \dot{\theta} \dot{\phi} \sin\theta \cos\theta \cos\phi \sin\phi$$

$$\dot{y}^2 = \dot{r}^2 \sin^2\theta \sin^2\phi + r^2 \dot{\theta}^2 \cos^2\theta \sin^2\phi + r^2 \dot{\phi}^2 \sin^2\theta \cos\phi + 2 r \dot{r} \dot{\theta} \sin\theta \cos\theta \sin^2\phi$$
$$+ 2 r \dot{r} \dot{\phi} \sin^2\theta \sin\phi \cos\phi + 2 r^2 \dot{\theta} \dot{\phi} \sin\theta \cos\theta \sin\phi \cos\phi$$

$$\dot{z}^2 = \dot{r}^2 \cos^2\theta - 2 r \dot{r} \dot{\theta} \sin\theta \cos\theta + r^2 \dot{\theta}^2 \sin^2\theta$$

Looking at the cross terms first, $\dot{r}\dot{\theta}$:

$$2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\sin^2\phi - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta$$

$$= 2r\dot{r}\dot{\theta}\sin\theta\cos\theta(\sin^2\phi + \cos^2\phi) - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta = 0$$

For $\dot{r}\dot{\phi}$ cross terms:

$$-2r\dot{r}\dot{\phi}\sin^2\theta\cos\phi\sin\phi + 2r\dot{r}\dot{\phi}\sin^2\theta\sin\phi\cos\phi = 0$$

For $\dot{\theta}\dot{\phi}$:

$$-2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\cos\phi\sin\phi + 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi = 0$$

The squared terms

$$\dot{r}^2\sin^2\theta\cos^2\phi + \dot{r}^2\sin^2\theta\sin^2\phi + \dot{r}^2\cos^2\theta = \dot{r}^2$$

$$r^2\dot{\theta}^2\cos^2\theta\cos^2\phi + r^2\dot{\theta}^2\cos^2\theta\sin^2\phi + r^2\dot{\theta}^2\sin^2\theta = r^2\dot{\theta}^2$$

$$r^2\dot{\phi}^2\sin^2\theta\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta\cos^2\phi = r^2\dot{\phi}^2\sin^2\theta$$

So $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r,\theta,\phi)$$

The momenta are: $P_r = \partial\mathcal{L}/\partial\dot{r} = m\dot{r}$, $P_\theta = \partial\mathcal{L}/\partial\dot{\theta} = mr^2\dot{\theta}$

and $P_\phi = \partial\mathcal{L}/\partial\dot{\phi} = mr^2\dot{\phi}\sin^2\theta$.

Inverting: $\dot{r} = \dfrac{P_r}{m}$, $\dot{\theta} = \dfrac{P_\theta}{mr^2}$, $\dot{\phi} = \dfrac{P_\phi}{mr^2\sin^2\theta}$.

The Hamiltonian becomes

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2\sin^2\theta} - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4} + r^2\sin^2\theta\frac{p_\phi^2}{m^2r^4\sin^4\theta}\right) + U$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2\sin^2\theta} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} - \frac{p_\phi^2}{2mr^2\sin^2\theta} + U$$

$$\boxed{= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r,\theta,\phi)}$$