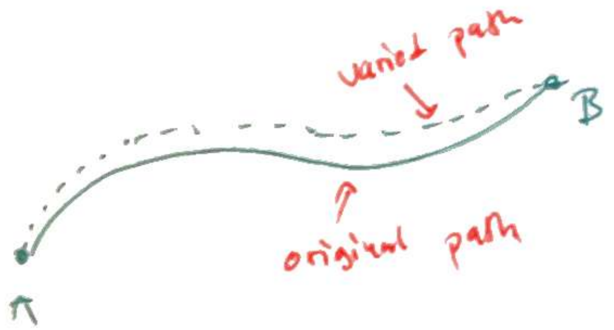
"Derive the differential equation of the path from the variational principle (44.10)."

The variational principle (44.10) is:

$$\delta \int \sqrt{[2m(E-U)]}\, dl = 0$$

The idea is to vary the path while holding the endpoints fixed.



Let $\bar{r} = (x, y, z)$ be the position in space and $U(r)$ the potential at that position. The question is: which path through space makes this integral stationary?

The integral has two parts that change when we nudge the path: 1. $f = \sqrt{2m(E-U)}$ and 2. dl.

When we shift a point on the path by a small displacement δr, the potential changes by: $\delta U = \nabla U \cdot \delta \bar{r}$. Using the chain rule we get:

$$\delta f = \delta \sqrt{2m(E-U)} = \frac{-m \nabla U \cdot \delta r}{\sqrt{2m(E-U)}} .$$

How does dl change when we nudge the path?

dl is a tiny piece of the curve, parametrized by r:
$dl = \sqrt{dr \cdot dr}$. When we nudge the path by δr:

$$dl \mapsto \sqrt{(dr + d(\delta r)) \cdot (dr + d(\delta r))} .$$

Expanding the above to first order in δr we drop the small $\delta r \cdot \delta r$ term and get:

$$\sqrt{dr \cdot dr + 2\, dr \cdot d(\delta r)} = dl \sqrt{1 + \frac{2 dr \cdot d(\delta r)}{dl^2}}$$

using our favorite approximation: $\sqrt{1+\varepsilon} \approx 1 + \frac{\varepsilon}{2}$ we have that:

$$\sqrt{1 + \frac{2 dr \cdot d(\delta r)}{dl^2}} \approx 1 + \frac{1}{2} \cdot \frac{2 dr \cdot d(\delta r)}{dl^2} = 1 + \frac{dr \cdot d(\delta r)}{dl^2}$$

The change in dl is therefore:

$$\delta(dl) = \frac{dr \cdot d(\delta r)}{dl} = \frac{dr}{dl} \cdot d(\delta r) = t \cdot d(\delta r)$$

where we have used $t = dr/dl$ as the unit tangent vector to the curve. The total variation is now

$$\delta S = \int_A^B \left[\delta f \cdot dl + f \cdot \delta(dl) \right]$$ where the first

term contributes $\displaystyle\int_A^B \delta f \, dl = - \int_A^B \frac{m \nabla u \cdot \delta r}{\sqrt{2m(E-u)}} \, dl$.

The second term is:

$$\int_A^B f \cdot \delta(d\ell) = \int_A^B ft \cdot d(\delta r) .$$

Here it is wise to use integration by parts to get rid of $d(\delta r)$. Let $u = ft$ and $dv = d(\delta r)$:

$$\int_A^B ft \cdot d(\delta r) = \Big[ft\,\delta r\Big]_A^B - \int_A^B \delta r \cdot d(ft)$$

As Ludwig Wittgenstein would put it, where variation cannot speak, it must remain silent: $\delta r = 0$ at A and B. So:

$$\int_A^B ft \cdot d(\delta r) = -\int_A^B \delta r \cdot d(ft) = -\int_A^B \frac{d(ft)}{d\ell} \cdot \delta r \, d\ell$$

Combining the effects:

$$\delta S = -\int_A^B \left[\frac{m \nabla u}{\sqrt{2m(E-u)}} + \frac{d(ft)}{d\ell}\right] \cdot \delta r \, d\ell = 0$$

Since δr is arbitrary at every point (interior), the bracket must vanish:

$$\frac{d(ft)}{d\ell} = -\frac{m \nabla u}{\sqrt{2m(E-u)}}$$ where the product rule

on LHS gives $\frac{df}{d\ell}t + f\frac{dt}{d\ell} = -\frac{m \nabla u}{\sqrt{2m(E-u)}}$.

where $d\hat{t}/dl = \hat{n}/R$ with $\hat{n}$ unit normal vector toward centre of curvature and R = radius of curvature.

Using this in our equation:

$$\frac{df}{dl}\hat{t} + \frac{f}{R}\hat{n} = -\frac{m\bar{\nabla}U}{\sqrt{2m(E-U)}}$$, dot both sides with $\hat{n}$:

$$\frac{df}{dl}(\hat{n}\cdot\hat{t}) + \frac{f}{R}(\hat{n}\cdot\hat{n}) = -\frac{m\bar{\nabla}U\cdot\hat{n}}{\sqrt{2m(E-U)}}$$ in which $\hat{n}\cdot\hat{t} = 0$

and $\hat{n}\cdot\hat{n} = 1$:

$$\frac{f}{R} = -\frac{m\bar{\nabla}U\cdot\hat{n}}{\sqrt{2m(E-U)}}$$

Energy conservation says $mv^2/2 = E - U$ so

$$f = mv \Rightarrow \frac{mv}{R} = -\frac{m\bar{\nabla}U\cdot\hat{n}}{mv} \Leftrightarrow \frac{mv^2}{R} = -\bar{\nabla}U\cdot\hat{n}$$

which is Newton's 2nd law for the normal force describing acceleration in curved motion.

The path equation is therefore consistent with the EOM, derived fully from the geometry of the variational principle.