Problem 4

"Find the potential of the electric field due to a small test charge e_1 at rest in the plasma."

so we have some static point charge sitting at rest. In vacuum it would produce a Coulomb potential $\varphi = e_1/r$, but in plasma, the particles rearrange around the point charge - partially cancelling the field.

The electric displacement $\bar{D}$ satisfies $\operatorname{div} \bar{D} = 4\pi\rho$. The only charge is the point charge, $\rho = e_1\delta(\bar{r})$ which is saying that the charge e_1 is concentrated at a single point.

$$\operatorname{div} \bar{D} = 4\pi e_1 \delta(\bar{r})$$

We will Fourier transform this equation because in real space, the "div" contains derivatives which are tricky to deal with. In Fourier space, $\bar{\nabla} \leftrightarrow i\bar{k}$, in other words, div becomes multiplication by $i\bar{k}$. Hence, $i\bar{k} \cdot \bar{D}_k = 4\pi e_1$.

The charge is at rest which means $\omega = 0$ and it produces a radial field which points along $\bar{k}$ in Fourier space. The field is longitudinal (radially outward from test charge), which means $\bar{D}_k = \varepsilon_\ell(0,k)\bar{E}_k$ where $\bar{E}_k = -i\bar{k}\varphi_k$ s.t.

$\bar{D}_k = -i\bar{k}\varepsilon_\ell(0,k)\varphi_k$. Now $i\bar{k}\cdot\bar{D}_k = i\bar{k}\cdot(-i\bar{k}\varepsilon_\ell(0,k)\varphi_k)$

$$\Rightarrow \varphi_k = \frac{4\pi e_1}{k^2 \varepsilon_\ell(0,k)}.$$

The value of $\varepsilon_\ell(0,k)$ can be obtained from:

$$\varepsilon_\ell - 1 = \frac{1}{(ka_e)^2} + \frac{1}{(ka_i)^2} + i\sqrt{\frac{\pi}{2}}\frac{\omega}{(ka_i)^2 k v_{Ti}}$$

which is the expression for permittivity when $\omega \ll k v_{Ti} \ll k v_{Te}$.
For us, given $\omega = 0$ and 1 = vacuum contribution:

$$\varepsilon_\ell(0,k) = 1 + \frac{1}{k^2 a_e^2} + \frac{1}{k^2 a_i^2} = 1 + \frac{1}{k^2 a^2}$$

where a is defined by $1/a^2 = 1/a_e^2 + 1/a_i^2$. Substituting:

$$\varphi_k = \frac{4\pi e_1}{k^2\left(1 + \frac{1}{k^2 a^2}\right)} = \frac{4\pi e_1}{k^2 + 1/a^2}$$. Time to get back

to real space, we need to invert the Fourier transform to get $\varphi(r)$:

$$\varphi(r) = \int \frac{4\pi e_1}{k^2 + 1/a^2} e^{i\bar{k}\cdot\bar{r}} \frac{d^3k}{(2\pi)^3}$$

Switching to spherical coordinates $d^3k = k^2 \sin\theta \, dk \, d\theta \, d\phi$ where θ is the angle between $\bar{k}$ and $\bar{r}$, so $\bar{k}\cdot\bar{r} = kr\cos\theta$.
The integrand does not depend on ϕ, so that integral $= 2\pi$:

$$\varphi(r) = \frac{4\pi e_1}{(2\pi)^3} \cdot 2\pi \int_0^\infty \int_0^\pi \frac{k^2}{k^2 + 1/a^2} e^{ikr\cos\theta} \sin\theta \, d\theta \, dk.$$

Let's do the θ integral first.

$$\int_0^{\pi} e^{ikr\cos\theta} \sin\theta \, d\theta = \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta \, d\theta \end{bmatrix}$$

$$= \int_{-1}^{1} e^{ikru} \, du = \left[\frac{e^{ikru}}{ikr} \right]_{-1}^{1} = \frac{e^{ikr} - e^{-ikr}}{ikr}$$

$$= \frac{2\sin(kr)}{kr}$$. Substituting back:

$$\varphi(r) = \frac{4\pi e_1}{(2\pi)^2} \int_0^{\infty} \frac{k^2}{k^2 + 1/a^2} \cdot \frac{2\sin(kr)}{kr} \, dk$$

$$= \frac{2 e_1}{\pi r} \int_0^{\infty} \frac{k \sin(kr)}{k^2 + 1/a^2} \, dk$$

Suppose $k\sin(kr) = \mathrm{Im}(k e^{ikr})$ and extend the integral to $(-\infty, \infty)$ with the idea that:

$$\int_0^{\infty} \frac{k \sin kr}{k^2 + 1/a^2} \, dk = \frac{1}{2} \mathrm{Im} \int_{-\infty}^{\infty} \frac{k e^{ikr}}{k^2 + 1/a^2} \, dk ,$$

The poles are at $k = \pm i/a$. For $r > 0$, enclosing only the pole at $+i/a$ by the residue theorem:

$$\int_{-\infty}^{\infty} \frac{k e^{ikr}}{(k - i/a)(k + i/a)} \, dk = 2\pi i \cdot \mathrm{Res}_{k = i/a}$$ where $\mathrm{Res}_{k=i/a}$

is $\lim\limits_{k \to i/a} (k - i/a) \dfrac{k e^{ikr}}{(k - i/a)(k + i/a)}$

The $(k - i/a)$ cancels such that $\mathrm{Res}_{k=i/a} = \lim_{k \to i/a} \frac{k e^{ikr}}{k+i/a}$

$$= \frac{(i/a) e^{i(i/a) r}}{(i/a) + (i/a)} = \frac{(i/a) e^{-r/a}}{2i/a} = \frac{e^{-r/a}}{2}.$$

So $2\pi i \cdot \mathrm{Res}_{k=i/a} = \pi i e^{-r/a}$. The imaginary part of this $\mathrm{Im}(\pi i e^{-r/a}) = \pi e^{-r/a}$ hence:

$$\int_0^\infty \frac{k \sin(kr)}{k^2 + 1/a^2} \, dk = \frac{\pi}{2} e^{-r/a}.$$

The final result is:

$$\varphi(r) = \frac{2 e_1}{\pi r} \cdot \frac{\pi}{2} e^{-r/a} = \boxed{\frac{e_1}{r} e^{-r/a}}$$

Interestingly, we got a Coulomb potential e_1/r multiplied with a decaying exponential. At $r \ll a$ we have $e^{-r/a} \approx 1$, the plasma hasn't screened the charge yet. At $r \sim a$ the potential drops below e_1/r and for $r \gg a$ $e^{-r/a} \approx 0$ the charge has ended up in heaven.

The "a" in $e^{-r/a}$ is the Debye length, i.e., the distance over which plasma particles rearrange themselves to screen the test charge. Even though Coulomb is long range, it is effectively cut off at the Debye length.