Problem

By "spatial oscillator" in 3D we mean that:

$\ddot{x} + \omega_0^2 x = 0$, $\ddot{y} + \omega_0^2 y = 0$ and $\ddot{z} + \omega_0^2 z = 0$.

Place this object in a magnetic field $\bar{H} = (0, 0, H)$

which produces a force $\bar{F}_L = \frac{e}{c} \bar{v} \times \bar{H}$ on the object.

The other force acting is determined by $\bar{F} = -\bar{\nabla} U(\bar{r})$

and with Taylor expansion this potential becomes

$U(x,y,z) = \frac{1}{2} m \omega_0^2 (x^2 + y^2 + z^2)$, hence $F_i = -m\omega_0^2 x_i$.

The equations of vibration (forced) become:

$$m\ddot{x} = -m\omega_0^2 x + \frac{eH}{c}\dot{y} \Rightarrow \ddot{x} + \omega_0^2 x = \frac{eH}{mc}\dot{y} \qquad (i)$$

$$m\ddot{y} = -m\omega_0^2 y + \left(-\frac{eH}{c}\dot{x}\right) \Rightarrow \ddot{y} + \omega_0^2 y = -\frac{eH}{mc}\dot{x} \qquad (ii)$$

$$m\ddot{z} = -m\omega_0^2 z \qquad \Rightarrow \ddot{z} + \omega_0^2 z = 0$$

Note that $\bar{v} \times \bar{H} = (\dot{y}H, -\dot{x}H, 0)$. The problem for us now is that these are coupled equations, so we need to untangle them. Define $\xi = x + iy$, then $\dot{\xi} = \dot{x} + i\dot{y}$ and $\ddot{\xi} = \ddot{x} + i\ddot{y}$. Take eq. (ii) and multiply by "i" and add it to eq (i):

$$(\ddot{x} + \omega_0^2 x) + i(\ddot{y} + \omega_0^2 y) = \frac{eH}{mc}\dot{y} - i\frac{eH}{mc}\dot{x}$$

Problem

We see that the LHS is $\ddot{z} + \omega_0^2 z$ and RHS is $\frac{eH}{mc}(\dot{y} - i\dot{x})$. Since $(\dot{y} - i\dot{x}) = -i(\dot{x} + i\dot{y}) = -i\dot{z}$

eq (i) becomes $\ddot{z} + \omega_0^2 z = -i \frac{eH}{mc} \dot{z}$.

Make an ansatz $z(t) = Ae^{-i\omega t}$ s.t. $\dot{z}(t) = -i\omega A e^{-i\omega t}$ and $\ddot{z}(t) = -\omega^2 A e^{-i\omega t}$. Insert into (i):

$$-\omega^2 Ae^{-i\omega t} + \omega_0^2 Ae^{-i\omega t} = -i \frac{eH}{mc}(-i\omega Ae^{-i\omega t})$$

$$\Rightarrow \quad \omega^2 - \frac{eH}{mc}\omega - \omega_0^2 = 0 \qquad \text{(quadratic eq. for } \omega\text{)}$$

has solutions
$$\omega = \frac{\frac{eH}{mc} \pm \sqrt{\left(\frac{eH}{mc}\right)^2 + 4\omega_0^2}}{2}$$

$$= \sqrt{\omega_0^2 + \frac{1}{4}\left(\frac{eH}{mc}\right)^2} \pm \frac{eH}{2mc}$$

What have we discovered? One mode rotates in the same direction as the H-field is driving and the other mode rotates against it. And perhaps this is a classical analog to QM phenomena like Zeeman splitting.