"The Hamiltonian of a system, and therefore the eigenvalues E_n of the energy, are functions of some parameter λ. Show that $(\partial H/\partial\lambda)_{nn} = \partial E_n/\partial\lambda$."

In other words, show $\langle n | \frac{\partial H}{\partial \lambda} | n \rangle = \frac{\partial E_n}{\partial \lambda}$. Starting from the eigenvalue equation $\hat{H}\psi_n = E_n \psi_n$ we have $(\hat{H} - E_n)\psi_n = 0$, indicating ψ_n is an eigenstate.

Differentiating: $\frac{\partial}{\partial\lambda}[(\hat{H} - E_n)\psi_n] = 0$

$$\Leftrightarrow (\partial_\lambda \hat{H})\psi_n - (\partial_\lambda E_n)\psi_n + (\hat{H} - E_n)\frac{\partial \psi_n}{\partial \lambda} = 0$$

Multiplying with ψ_n^*: $\psi_n^*(\hat{H} - E_n)\frac{\partial \psi_n}{\partial\lambda} = \psi_n^*\left(\frac{\partial E_n}{\partial\lambda} - \frac{\partial \hat{H}}{\partial\lambda}\right)\psi_n$

and integrating both sides over all space q

$$\int \psi_n^* (\hat{H} - E_n)\frac{\partial\psi_n}{\partial\lambda}\, dq = \int \psi_n^* \left(\frac{\partial E_n}{\partial\lambda} - \frac{\partial\hat{H}}{\partial\lambda}\right)\psi_n\, dq \quad (i)$$

Using the hermiticity of $\hat{H}$ on LHS we see that

$$\int \psi_n^* (\hat{H} - E_n)\frac{\partial\psi_n}{\partial\lambda}\, dq = \int \frac{\partial\psi_n}{\partial\lambda}(\hat{H} - E_n)\psi_n^*\, dq = 0$$

since $(\hat{H} - E_n)\psi = 0 \Rightarrow (\hat{H} - E_n)\psi^* = 0$. The RHS of (i)

$$\int \psi_n^*\left(\frac{\partial E_n}{\partial\lambda} - \frac{\partial\hat{H}}{\partial\lambda}\right)\psi_n\, dq = 0$$ and splitting terms:

$$\frac{\partial E_n}{\partial\lambda}\int |\psi_n|^2\, dq - \int \psi_n^* \frac{\partial\hat{H}}{\partial\lambda}\psi_n\, dq = 0$$

Leaves us with $\frac{\partial E}{\partial \lambda} = \int \psi_n^* \frac{\partial \hat{H}}{\partial \lambda} \psi_n \, dq$

or in other words:

$$\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial \hat{H}}{\partial \lambda} | \psi_n \rangle \qquad \square$$

The important thing to take away is the fact that if we tweak a parameter in the Hamiltonian then the change in energy will be the average of how ~~that~~ the Hamiltonian changes in that state.