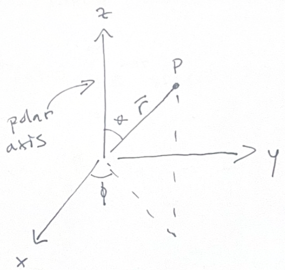
PROBLEM 1.

"Determine the probability distribution of various values of the momentum in ...."

The wave function in the "$\bar{p}$ representation" is given by $a(\bar{p}) = (2\pi\hbar)^{-3/2} \int \psi(\bar{r})\, e^{-(i/\hbar)\bar{p}\cdot\bar{r}}\, dV.$

The ground state is $\psi_{100}(r,\theta,\phi) = R_{10}(r)\, Y_{00}(\theta,\phi)$ $= \frac{1}{\sqrt{\pi}} e^{-r}$. Furthermore it is convenient to set $\hbar = 1$

s.t. $a(\bar{p}) = (2\pi)^{-3/2} \int \psi(\bar{r})\, e^{-i\bar{p}\cdot\bar{r}}\, dV.$



The problem already suggested spherical coordinates, best if we let the polar axis be chosen along $\bar{p}$ s.t. $\bar{p}\cdot\bar{r} = pr\cos\theta$.

$$\Rightarrow a(\bar{p}) = \frac{1}{(2\pi)^{3/2}\sqrt{\pi}} \int_0^{2\pi}\int_0^{\infty}\int_0^{\pi} e^{-r} e^{-ipr\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Let $I = \int_0^{\pi} e^{-ipr\cos\theta} \sin\theta\, d\theta = \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{bmatrix}$

$$= -\int_1^{-1} e^{-ipru}\, du = \frac{e^{ipr} - e^{-ipr}}{ipr} = \frac{2\sin(pr)}{pr}$$

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We have $$a(p) = \frac{2\pi}{(2\pi)^{3/2}\sqrt{\pi}} \int_0^\infty e^{-r} r^2 \frac{2\sin(pr)}{pr}\, dr$$

$$= \frac{4\pi}{(2\pi)^{3/2}\sqrt{\pi}\, p} \int_0^\infty r e^{-r} \sin(pr)\, dr$$

Let $F(a) = \int_0^\infty e^{-ar} \sin(pr)\, dr$ which means

$\int_0^\infty r e^{-r} \sin(pr)\, dr = -\frac{d}{da} F(a)\Big|_{a=1}$. $F(a)$ can be solved multiple ways, let's Laplace it:

$$F(a) = \int_0^\infty e^{-ar} \sin(pr)\, dr = \mathcal{L}\{\sin(pr)\}(a) = \frac{p}{a^2+p^2}$$

and $$-\frac{d}{da} F(a)\Big|_{a=1} = -\left(-\frac{2ap}{(a^2+p^2)^2}\right)\Big|_{a=1} = \frac{2p}{(1+p^2)^2}.$$

Then $$a(p) = \frac{4\pi}{(2\pi)^{3/2}\sqrt{\pi}\, p} \cdot \frac{2p}{(1+p^2)^2} = \frac{8}{2^{3/2}\pi} \cdot \frac{1}{(1+p^2)^2} = \frac{2\sqrt{2}}{\pi(1+p^2)^2}$$

with probability ~~distribution~~ density $\boxed{|a(p)|^2 = \frac{8}{\pi^2(1+p^2)^4}}$.