Problem 7.

"Find the mean value of the nth power of the absolute magnitude of the velocity."

The mean of any quantity is that quantity weighted by its probability and integrated over all possible values:

$$\overline{v^n} = \int_0^\infty v^n \, dw_v$$ where dw_v is the probability

of finding the particle with speed between v and $v+dv$. The definition of dw_v is given as:

$$dw_v = 4\pi \left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T} v^2 \, dv.$$ So we get that

$$\overline{v^n} = \int_0^\infty v^n \cdot 4\pi \left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T} v^2 \, dv$$

$$= 4\pi \left(\frac{m}{2\pi T}\right)^{3/2} \int_0^\infty e^{-mv^2/2T} v^{n+2} \, dv$$

This integral is the form of a standard one

$$I_n = \int_0^\infty e^{-\alpha x^2} x^n \, dx$$ with $\alpha = m/2T$ whose result is

$$I_n = \frac{1}{2} \alpha^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right)$$ which for us with I_{n+2} is

$$\int_0^\infty e^{-mv^2/2T} v^{n+2} \, dv = \frac{1}{2}\left(\frac{m}{2T}\right)^{-(n+3)/2} \Gamma\left(\frac{n+3}{2}\right)$$

Hence $\overline{v^n} = 4\pi \left(\frac{m}{2\pi T}\right)^{3/2} \cdot \frac{1}{2}\left(\frac{m}{2T}\right)^{-(n+3)/2} \Gamma\left(\frac{n+3}{2}\right)$

$$\boxed{= \frac{2}{\sqrt{\pi}}\left(\frac{2T}{m}\right)^{n/2} \Gamma\left(\frac{n+3}{2}\right)}$$

Some special cases are when n is even or odd. If $n = 2r$, $r \in \mathbb{Z}^+$, then $\frac{n+3}{2} = r + \frac{3}{2}$. The recursion formula $\Gamma(x+1) = x\Gamma(x)$ yields:

$\Gamma(r+\frac{3}{2}) = (r+\frac{1}{2})(r-\frac{1}{2}) \dots \frac{1}{2}\sqrt{\pi}$ where we are stepping down to $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Now, factor out $\frac{1}{2}$ from every term in the product, there are $r+1$ terms.

$$(r+\tfrac{1}{2})(r-\tfrac{1}{2}) \dots \tfrac{1}{2}\sqrt{\pi} = \frac{1}{2^{r+1}}(2r+1)(2r-1)\dots 1 \cdot \sqrt{\pi}$$

$$= \frac{(2r+1)!!}{2^{r+1}}\sqrt{\pi}$$

where $(2r+1)!!$ is just the product of all odd numbers down to 1.

$$\boxed{\overline{v^{2r}} = \frac{2}{\sqrt{\pi}}\left(\frac{2T}{m}\right)^{r} \cdot \frac{(2r+1)!!}{2^{r+1}}\sqrt{\pi} = \left(\frac{T}{m}\right)^{r}(2r+1)!!}$$

If n instead is odd, $n = 2r+1$, then $\frac{n+3}{2} = r+2$. so $\Gamma(r+2) = (r+1)!$ substituting this into $\overline{v^n}$:

$$\boxed{\overline{v^{2r+1}} = \frac{2}{\sqrt{\pi}}\left(\frac{2T}{m}\right)^{(2r+1)/2}(r+1)!}$$