Problem 5

"Find the probability distribution for the angular velocities of rotation of molecules."

The rotational energy is $\varepsilon = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$

The classical Gibbs distribution gives the probability density in phase space as $\rho(p,q) = A e^{-E(p,q)/T}$. The probability of finding the system in a region of phase space is

$$dw = \rho(p,q)\,dp\,dq = A e^{-\varepsilon/T}\,dp_1\,dp_2\,dp_3\,dq_1\,dq_2\,dq_3 .$$

Since ε does not depend on q_i and $p_i = I_i\Omega_i$ we get

$$dw_\Omega = A(I_1I_2I_3)\exp\left[-\frac{1}{2T}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)\right] d\Omega_1\,d\Omega_2\,d\Omega_3 .$$

We require $\int dw_\Omega = 1$ and using $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\pi/\alpha}$ with $\alpha_i = I_i/2T$ we get that provided A contains I_i and the constant of integration from dq_i:

$$A\cdot\sqrt{\frac{2\pi T}{I_1}}\cdot\sqrt{\frac{2\pi T}{I_2}}\cdot\sqrt{\frac{2\pi T}{I_3}} = 1 \Rightarrow A = (2\pi T)^{-3/2}(I_1I_2I_3)^{1/2}$$

giving us:

$$\boxed{dw_\Omega = (2\pi T)^{-3/2}(I_1I_2I_3)^{1/2}\exp\left[-\frac{1}{2T}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)\right] d\Omega_1\,d\Omega_2\,d\Omega_3}$$