Problem 6

"Find the mean squares of the absolute magnitudes of the angular velocity and angular momentum of a molecule."

The mean of any quantity f is given by $\bar{f} = \int f \, dw$.

In our case, we want the mean of $\Omega^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2$

where $dw_\Omega = (2\pi T)^{-3/2} (I_1 I_2 I_3)^{1/2} \exp\left[-\frac{1}{2T}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)\right] d\Omega_1 \, d\Omega_2 \, d\Omega_3$

s.t., $\overline{\Omega^2} = \int (\Omega_1^2 + \Omega_2^2 + \Omega_3^2) \, dw_\Omega = \overline{\Omega_1^2} + \overline{\Omega_2^2} + \overline{\Omega_3^2}$

The integral splits into 3 terms where for each i:

$$\overline{\Omega_i^2} = \sqrt{\frac{I_i}{2\pi T}} \int_{-\infty}^{\infty} \Omega_i^2 e^{-I_i \Omega_i^2 / 2T} \, d\Omega_i \quad = \left[\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \right]$$

$$= \sqrt{\frac{I_i}{2\pi T}} \cdot \frac{\sqrt{\pi}}{2\left(\frac{I_i}{2T}\right)^{3/2}} = \frac{2^{3/2} \, T}{2 \cdot 2^{1/2} \cdot I_i} = \frac{T}{I_i}.$$

Since each term gives $\frac{T}{I_i}$, $\overline{\Omega^2} = \boxed{T\left[\frac{1}{I_1} + \frac{1}{I_2} + \frac{1}{I_3}\right]}$

Note that $M_i = I_i \Omega_i$ and I_i constant:

$$\overline{M_i^2} = \overline{I_i^2 \Omega_i^2} = I_i^2 \overline{\Omega_i^2} = I_i^2 \cdot \frac{T}{I_i} = I_i T \quad \text{so}$$

$$\boxed{\overline{M^2} = T(I_1 + I_2 + I_3)}$$