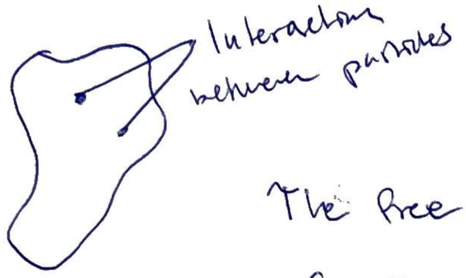
Problem 7.

"The potential energy of the interaction between the particles in a body is a homogeneous function ..."



The free energy is $F = -T \log Z$ where

$$Z = \int e^{-[K(p) + U(q)]/T} \, d\Gamma$$

where the volume element

$d\Gamma = dp\,dq/(2\pi\hbar)^s$. If we scale $q_i \mapsto \lambda q_i$, $p_i \mapsto \lambda^{n/2} p_i$ and $T \mapsto \lambda^n T$. By assumption $U(q)$ is homogeneous degree n, so $U(\lambda q) = \lambda^n U(q)$. For $K(p)$, note that K is degree 2 in p: $K(\lambda^{n/2} p) = (\lambda^{n/2})^2 K(p) = \lambda^n K(p)$. The differentials dq_i and dp_i scale as:

$$\begin{cases} dq_i \mapsto \lambda\, dq_i \text{ (with 3N of them) becomes } dq \to \lambda^{3N} dq \\ dp_i \mapsto \lambda^{n/2} dp_i \text{ (with 3N of them) becomes } dp \mapsto \lambda^{3Nn/2} dp \end{cases}$$

The region of integration has each length scale by λ so $V \mapsto \lambda^3 V$. The partition function becomes:

$$Z(V,T) \to \lambda^{3N(1+n/2)} \cdot Z(\lambda^3 V, \lambda^n T) \quad \text{because}$$

$d\Gamma \to \lambda^{3N} \cdot \lambda^{3Nn/2}\, d\Gamma = \lambda^{3N(1+n/2)}\, d\Gamma$. Now, we want to find the most general function $Z(V,T)$ s.t. when $V \to \lambda^3 V$ and $T \mapsto \lambda^n T$, the function scales by $\lambda^{3N(1+n/2)}$.

It makes sense to try $Z(V,T) = T^\alpha f(VT^\beta)$

where α & β are numbers and f arbitrary function.

$$Z(\lambda^3 V, \lambda^4 T) = (\lambda^4 T)^\alpha f(\lambda^3 V \cdot (\lambda^4 T)^\beta)$$

$$= \lambda^{4\alpha} T^\alpha f(\lambda^{3+4\beta} V T^\beta)$$

For this to equal $\lambda^{-3N(1+4/2)} Z(V,T)$ we need

$\lambda^{4\alpha} = \lambda^{-3N(1+4/2)} \Rightarrow 4\alpha = -3N(1+4/2) \Rightarrow \alpha = -3N\left(\frac{1}{2}+\frac{1}{4}\right)$

and $f(\lambda^{3+4\beta} V T^\beta) = f(V T^\beta)$ becomes $3+4\beta = 0$

$\Rightarrow \beta = -3/4$. Hence we can write:

$$Z(V,T) = T^{-3N(1/2+1/4)} f(V T^{-3/4}) \text{ and } F \text{ becomes}$$

$$F = -T\log Z = 3N\left(\frac{1}{2}+\frac{1}{4}\right) T\log T - T\log f(V T^{-3/4}).$$

Now, we want F to be extensive, which is true for the first term since it contains N but $-T\log f(VT^{-3/4})$ is not. To make sure it scales properly when N and V double we are forced to write $NTf(VT^{-3/4}/N)$:

$$F = -3\left(\frac{1}{2}+\frac{1}{4}\right) NT\log T + NT\phi(VT^{-3/4}/N)$$

(where $\phi = -\frac{1}{N}\log f$)