1.1.1 Problem 1

Determine the shape of the surface of an incompressible fluid subject to a gravitational field, contained in a cylindrical vessel which rotates about its (vertical) axis with a constant angular velocity Ω.

Incompressible means that the fluid cannot be squeezed into a smaller space, or stretched into a bigger one, i.e., the volume stays the same throughout. The fixed quantities are gravity, angular velocity and volume. What type of answer are we looking for? It is a shape. Think about stirring coffee in a mug ... What happens to the surface? Well it dips down in the middle and rises at the rim. Also, it looks the same no matter what direction around the axis you look. This suggests that the shape is some curve, in other words, height plotted against how far out the center one is, and then spun around the axis to get the full 3D shape.

What would make the answer obviously wrong? E.g., if you spin it around twice as fast, would you imagine that the bowl gets twice as deep or something else? Think of spinning a ball on a rope and doubling the angular velocity, do you have to hold it twice as hard or even harder? The answer is that doubling leads to a quadrupling. The centrifugal goes as the square of the angular rate.

Now is there any analogy you can find to this problem? The spinning fluid is pushing everything outward from the center while gravity is pulling everything down. This is very reminiscent of a car going around a corner on a banked road.

Estimate the answer in your head. You already know that it involves a shape $\propto \Omega^2$, but it also has to involve gravity somehow since the fluid is fighting it (aren't we all fighting it). Thinking back to the banked road car example, if gravity was stronger, would the road need to be more steep or less steep? Ok, so clearly less steep. Notice how this matches the fluid problem, bigger g means flatter surface so the shape is $\propto \Omega^2/g$. Suppose the distance from the center is described by the radius vector $\boldsymbol{r}$, the height then would be something like $\Omega^2\boldsymbol{r}^2/g$ give or take some prefactor.

How is the fluid moving? How fast is a fluid particle moving if it sits at distance $\boldsymbol{r}$ from the center? The answer is $\boldsymbol{\Omega} \times \boldsymbol{r}$. The further out we are on the roundabout the faster it goes. Now, the physics of the shape is such that every point is pulled down and flung outward, a balance if you will. If I want to go in a circle then I need to accelerate with v^2/r, substituting $v = \Omega r$ then the acceleration is $\Omega^2 r$.

Considering the outward acceleration is $\Omega^2 r$ and downward is g, the surface has to tilt at every point such that $dz/dr = \Omega^2 r/g \implies z(r) = \Omega r^2/2g$ where z is the height of the surface. That's it, a friendly paraboloid.