§2

The point of this exercise is to learn how to rewrite the fundamental fluid equations in Lagrangian variables.

The book has given us the fluid EOM in Eulerian form of functions of x, t. This problem asks us to rewrite them using Lagrangian variables a & t.

We want $\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ (i) to be written in/with particle label a.

Let the particle position be $x = x(a,t)$ where a is the position of the particle at t_0. Simply we can see $v(a,t) = (\partial x/\partial t)_a$ and $dv/dt = (\partial v/\partial t)_a$. The next step is to deal with the RHS of (i), rather how do we convert $\partial/\partial x \mapsto \partial/\partial a$?

Well, we need to figure out how x changes with a. Start from the continuity equation

$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$ which tells us that

the total mass does not change if we take a group of fluid particles and follow them as they move. Take some particles that lie between a & $a + da$, then $dm = \rho_0(a)\, da$ similarly with letting these occupy $[x, x+dx]$, i.e., $dm = \rho(x,t)\, dx$.

Since they are the same particles, continuity says that $\rho\, dx = \rho_0\, da$. Our next task is to remove dx.

Since $x = x(a,t)$ then $dx = \left(\frac{\partial x}{\partial a}\right) da$

$$\Rightarrow \rho \left(\frac{\partial x}{\partial a}\right)_t da = \rho_0\, da$$

$$\Leftrightarrow \rho \left(\frac{\partial x}{\partial a}\right)_t = \rho_0 \quad \text{(ii)}$$

In the RHS of the (i) eq. we have $\frac{1}{\rho} \frac{\partial p}{\partial x}$

and $\frac{1}{\rho} = \frac{1}{\rho_0}\left(\frac{\partial x}{\partial a}\right)_t$ from (ii). Using the chain rule on p: $\left(\frac{\partial p}{\partial a}\right)_t = \left(\frac{\partial p}{\partial x}\right)_t \left(\frac{\partial x}{\partial a}\right)_t$

$$\Rightarrow \left(\frac{\partial p}{\partial x}\right)_t = \frac{\left(\frac{\partial p}{\partial a}\right)_t}{\left(\frac{\partial x}{\partial a}\right)_t} \Rightarrow \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_t = \frac{1}{\rho_0}\left(\frac{\partial p}{\partial a}\right)_t$$

Hence the Euler equation becomes:

$$\left(\frac{\partial v}{\partial t}\right)_a = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial a}\right)_t$$