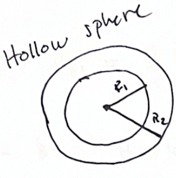
Problem II



The symmetry of the object suggests spherical symmetry.
When deformation is caused by forces applied to the surface of the object, the equilibrium equation becomes $(1-2\sigma)\Delta\bar{u} + \overline{\text{grad}}\ \text{div}\ \bar{u} = 0$.

Since the geometry is spherical, the displacement vector $\bar{u}$ must be purely radial. A consequence of this is the fact that nothing twists around the center, i.e., $\bar{\nabla} \times \bar{u} = 0$. Making use of the following vector identity: $\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = \Delta\bar{u} + \bar{\nabla}\times(\bar{\nabla}\times\bar{u})$ and $\bar{\nabla}\times\bar{u} = 0$ gives us $\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = \Delta\bar{u}$. Substituting this into the E.E. we get $(1-2\sigma)\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) + \bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = 0$

$\Rightarrow (2-2\sigma)\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = 0 \Rightarrow \overline{\text{grad}}\ \text{div}\ \bar{u} = 0$, $[\sigma \neq 1]$.

This means that $\text{div}\ \bar{u}$ = constant. Specifically,

$$\text{div}\ \bar{u} = \frac{1}{r^2}\frac{d}{dr}(r^2 u) = C \Rightarrow u(r) = \frac{r}{3}C + \frac{b}{r^2}.$$

As you can see, best if we can let $C = 3a$ since $u(r) = ar + \frac{b}{r^2}$. The strain components considering u only depends on r becomes $u_{rr} = \frac{\partial u}{\partial r}$ &
$u_{\theta\theta} = u_{\phi\phi} = \frac{u}{r}$ (see eq. 1.7 in §1.)

Problem II

$$u_{rr} = \frac{d}{dr}\left[ar + \frac{b}{r^2}\right] = a - \frac{2b}{r^3}$$

$$u_{\theta\theta} = u_{\phi\phi} = \frac{1}{r}\left[ar + \frac{b}{r^2}\right] = a + \frac{b}{r^3}$$

From §4. Hooke's law for an isotropic solid is

$\sigma_{ik} = K u_{ll}\delta_{ik} + 2\mu(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik})$ where u_{ll} is the trace, $u_{ll} = u_{rr} + u_{\theta\theta} + u_{\phi\phi}$. This represents the total volume strain:

$$u_{ll} = a - \frac{2b}{r^3} + 2\left[a + \frac{b}{r^3}\right] \boxed{= 3a}$$

Now, apply Hooke's law to get σ_{rr}, so for $i=k=r$

$\sigma_{rr} = K(3a) + 2\mu(-\frac{2b}{r^3})$ where we let $u_{ll} = 3a$ & $u_{rr} = a - \frac{2b}{r^3}$. Simplified, $\sigma_{rr} = 3Ka - \frac{4\mu b}{r^3}$. Typically we use E & σ instead of K & μ so $\mu = E/2(1+\sigma)$ and $K = E/3(1-2\sigma)$. Granted, we get that:

$$\sigma_{rr} = \frac{E}{1-2b}a - \frac{2E}{1+\sigma}\frac{b}{r^3}$$

To find a & b we need the boundary conditions $\sigma_{rr}(R_1) = -P_1$ & $\sigma_{rr}(R_2) = -P_2$. (The pressure is causing compression directed inward opposite the normal of the surface.)

Problem II

$$\sigma_{rr}(R_1) = \frac{E}{1-2\sigma} a - \frac{2E}{1+\sigma} \frac{b}{R_1^3} = -P_1 \quad (i)$$

$$\sigma_{rr}(R_2) = \frac{E}{1-2\sigma} a - \frac{2E}{1+\sigma} \frac{b}{R_2^3} = -P_2 \quad (ii)$$

Subtract (i) with (ii):

$$-\frac{2E}{1+\sigma}\left[\frac{b}{R_1^3} - \frac{b}{R_2^3}\right] = -P_1 + P_2$$

$$\Rightarrow \quad b = \frac{1+\sigma}{2E} \frac{R_1^3 R_2^3 (P_1 - P_2)}{R_2^3 - R_1^3}$$

Finding a from (i): $\frac{E}{1-2\sigma} a - \frac{2E}{1+\sigma} \frac{1}{R_1^3}\left[\frac{1+\sigma}{2E} \frac{R_1^3 R_2^3 (P_1 - P_2)}{R_2^3 - R_1^3}\right] = -P_1$

$$\Leftrightarrow \quad a = \frac{1}{E}\left[-P_1 + \frac{R_2^3 (P_1 - P_2)}{R_2^3 - R_1^3}\right](1-2\sigma)$$

$$\Leftrightarrow \quad a = \frac{1-2\sigma}{E}\left[\frac{P_1 R_2^3 - P_2 R_2^3}{R_2^3 - R_1^3}\right].$$ The final expression

for $\bar{u}$ becomes:

$$\bar{u}(r) = \left[\frac{1-2\sigma}{E} \frac{P_1 R_2^3 - P_2 R_2^3}{R_2^3 - R_1^3} r + \frac{1+\sigma}{2E} \frac{R_1^3 R_2^3 (P_1 - P_2)}{R_2^3 - R_1^3} \frac{1}{r^2}\right] \hat{r}$$