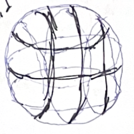
Problem III

The equations of equilibrium for a uniform gravitational field is:

$$\overline{\text{grad}}\ \text{div}\ \bar{u} - \frac{1-2\sigma}{2(1-\sigma)}\ \overline{\text{curl}}\ \overline{\text{curl}}\ \bar{u} = -\rho\bar{g}\,\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}$$

Solid sphere:



As in problem II, the spherical symmetry leads to $\overline{\text{curl}}\ \bar{u} = 0$ since $\bar{u}$ only depends on radial coordinate.

The E.E. then becomes $\overline{\text{grad}}\ \text{div}\ \bar{u} = -\rho\bar{g}\,\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}$.

Let $\bar{g} = g\frac{\bar{r}}{R}$ such that gravity is zero at the surface and increases linearly. Also recall $\overline{\text{grad}}\ \text{div}\ \bar{u} = \hat{r}\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)$.

To reduce the "crow," set $A = \frac{\rho g}{R}\,\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}$ s.t.,

$$\hat{r}\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = A\bar{r}$$, noting that $\hat{r} = \frac{\bar{r}}{|\bar{r}|}$ we get:

$$\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = Ar.$$ Integrating twice yields:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = \frac{Ar^2}{2} + C \Rightarrow u(r) = \frac{A}{10}r^3 + \frac{C}{3}r + \frac{D}{r^2}$$

Finitness requires $D = 0$, so $u(r) = \frac{A}{10}r^3 + \frac{C}{3}r$. For simplicity let $A' = \frac{A}{10}$ and $C' = \frac{C}{3}$ s.t. $u(r) = A'r^3 + C'r$.

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The strain formulas for spherical coordinates:

$$u_{rr} = \frac{du}{dr}, \quad u_{\theta\theta} = u_{\phi\phi} = \frac{u}{r}.$$ Computing these gives

us $u_{rr} = \frac{d}{dr}\left[A'r^3 + c'r\right] = 3A'r^2 + c'$ and

$u_{\theta\theta} = u_{\phi\phi} = \frac{1}{r}\left[A'r^3 + c'r\right] = A'r^2 + c'$. The stress tensor

is $\sigma_{ik} = K u_{ll}\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right)$, where u_{ll} = trace,

$K = E/3(1-2\sigma)$ and $\mu = E/2(1+\sigma)$. Hence we get

$$u_{ll} = u_{rr} + u_{\theta\theta} + u_{\phi\phi} = 3A'r^2 + c' + 2A'r^2 + 2c' = 5A'r^2 + 3c'.$$

$$\sigma_{rr} = \frac{E}{3(1-2\sigma)}\left[5A'r^2 + 3c'\right] + \frac{E}{(1+\sigma)}\left[3A'r^2 + c' - \frac{1}{3}(5A'r^2 + 3c')\right]$$

$$= \frac{E}{3(1-2\sigma)}\left[5A'r^2 + 3c'\right] + \frac{E}{1+\sigma}\left[\frac{4}{3}A'r^2\right]$$

$$= \frac{E}{(1+\sigma)(1-2\sigma)}\left[(1+\sigma)c' + (3-\sigma)A'r^2\right].$$

By the problem statement we have no external stress acting on the surface, so $\sigma_{rr}(R) = 0$ which gives

$$(1+\sigma)c' + (3-\sigma)A'R^2 = 0 \Rightarrow c' = -\frac{3-\sigma}{1+\sigma}A'R^2.$$

The displacement becomes $u(r) = A'r^3 - \frac{3-\sigma}{1+\sigma}A'R^2 r$

and let's massage this a bit.

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Remember that $A' = \frac{A}{10}$ where $A = \frac{\rho g}{R} \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}$.

Hence, $u(r) = A'\left(r^3 - \frac{3-\sigma}{1+\sigma} R^2 r\right)$

$$= \frac{\rho g}{10RE} \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)} \left[r^2 - \frac{3-\sigma}{1+\sigma} R^2\right] r$$

$$= \frac{\rho g (1+\sigma)(1-2\sigma)}{10E(1-\sigma)} \left[\frac{r^2}{R} - \frac{3-\sigma}{1+\sigma}\right]$$

$\square$

We have learned that stress is maximum at the center and vanishes at the surface of the solid sphere.