Problem IV

This problem is similar to problem II, but here we will use cylindrical coordinates instead.

Let the displacement vector be $\bar{u} = (u_r, u_\phi, u_z)$. Now, nothing changes if we rotate the system around the z-axis, nor does the pressure change with z, ~~and~~ so we are only left with $\bar{u} = (u_r, 0, 0)$.

This means $\overline{\text{grad}}\,\text{div}\,\bar{u} = 0 \Rightarrow \text{div}\,\bar{u} = \text{constant} := 2a$.

From cylindrical coordinates r, ϕ, z:

$$u_{rr} = \frac{\partial u_r}{\partial r}, \quad u_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}, \quad u_{zz} = \frac{\partial u_z}{\partial z}$$

$$\Rightarrow u_{rr} = \frac{\partial u}{\partial r}, \quad u_{\phi\phi} = \frac{u}{r} \quad \& \quad u_{zz} = 0$$

where u is obtained by $\frac{1}{r}\frac{d(ru)}{dr} = 2a \Rightarrow u(r) = ar + \frac{b}{r}$.

yielding $u_{rr} = a - \frac{b}{r^2}$ and $u_{\phi\phi} = a + \frac{b}{r^2}$. As customary by now, the stress tensor is

$$\sigma_{ik} = K u_{ll}\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}u_{ll}\delta_{ik}\right) \quad \text{where} \quad K = E/3(1-2\sigma)$$

and $\mu = E/2(1+\sigma)$

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$$u_{ll} = a - \frac{b}{r^2} + a + \frac{b}{r^2} = 2a.$$

$$\sigma_{rr} = \frac{E}{3(1-2\sigma)}(2a) + \frac{E}{1+\sigma}\left(a - \frac{b}{r^2} - \frac{1}{3}2a\right)$$

$$= \frac{E}{3(1+\sigma)(1-2\sigma)}\left[2a(1+\sigma) + a(1-2\sigma) - \frac{3b(1-2\sigma)}{r^2}\right]$$

$$= \frac{E}{(1+\sigma)(1-2\sigma)}\left[a - \frac{(1-2\sigma)b}{r^2}\right]$$

From the boundary conditions $\sigma_{rr} = 0$ at $r = R_2$ and $\sigma_{rr} = -p$ at $r = R_1$ we get:

$$a - \frac{(1-2\sigma)b}{R_2^2} = 0 \qquad \text{(i)}$$

$$\frac{E}{(1+\sigma)(1-2\sigma)}\left[a - \frac{(1-2\sigma)b}{R_1^2}\right] = -p \qquad \text{(ii)}$$

Insert a from (i) into (ii):

$$\frac{E}{(1+\sigma)(1-2\sigma)}\left[\frac{(1-2\sigma)b}{R_2^2} - \frac{(1-2\sigma)b}{R_1^2}\right] = -p$$

$$\Leftrightarrow \quad b\left[\frac{1}{R_2^2} - \frac{1}{R_1^2}\right] = -\frac{p(1+\sigma)}{E}$$

$$\Leftrightarrow \quad b = \frac{pR_1^2R_2^2}{R_2^2 - R_1^2}\cdot\frac{1+\sigma}{E}.$$

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Substituting back b in eq. (i):

$$a = \frac{(1-2\sigma)}{R_2^2} \cdot \frac{pR_1^2R_2^2}{R_2^2-R_1^2} \cdot \frac{1+\sigma}{E}$$

$$= \frac{pR_1^2}{R_2^2-R_1^2} \frac{(1+\sigma)(1-2\sigma)}{E} .$$

The final answer for the displacement vector is

$\bar{u}(r) = (ar + \frac{b}{r^2})\,\hat{r}$ with $a = \frac{pR_1^2}{R_2^2-R_1^2} \frac{(1+\sigma)(1-2\sigma)}{E}$

and $b = \frac{pR_1^2R_2^2}{R_2^2-R_1^2} \frac{1+\sigma}{E}$. With corresponding

stresses σ_{rr}, $\sigma_{\varphi\varphi}$ and σ_{zz} as:

$$\sigma_{rr} = \frac{pR_1^2}{R_2^2-R_1^2}\left(1 - \frac{R_2^2}{r^2}\right), \quad \sigma_{\varphi\varphi} = \frac{pR_1^2}{R_2^2-R_1^2}\left(1 + \frac{R_2^2}{r^2}\right)$$

and $\sigma_{zz} = \frac{2p\sigma R_1^2}{R_2^2-R_1^2}$

(It is optional to display the stresses, stating your result for $\bar{u}$ is sufficient.)