1431. $\displaystyle\int \frac{dx}{2x^2 - 4x + 9}$

For an integral involving $ax^2 + bx + c$, Demidovich says the principal procedure is to reduce it to

$$a(x+k)^2 + l$$

by taking out the perfect square. Starting with $2x^2 - 4x + 9 = 2(x^2 - 2x) + 9$ where $x^2 - 2x = (x-1)^2 - 1$ so the integral becomes

$$I = \int \frac{dx}{2(x-1)^2 + 7}, \quad \text{we want the tabular}$$

integral $\int \frac{dt}{1+t^2} = \arctan t + C$. Factoring out 7 from the denominator $2(x-1)^2 + 7 = 7\left[1 + \frac{2}{7}(x-1)^2\right]$ where we can set $t = \sqrt{\frac{2}{7}}(x-1)$ so $dt = \sqrt{2/7}\, dx$. Thus,

$$I = \int \frac{\sqrt{7/2}\, dt}{7(1+t^2)} = \frac{1}{\sqrt{14}} \int \frac{dt}{1+t^2} = \frac{1}{\sqrt{14}} \arctan t + C.$$

The thing to retain is $\frac{\text{constant}}{\text{quadratic}}$ $\rightarrow$ complete the square first,

1432. $\displaystyle\int \frac{x-5}{x^2-2x+2}\,dx$

This is one of the standard integrals $\displaystyle\int \frac{mx+n}{ax^2+bx+c}\,dx$

where Demidovich says to rewrite the numerator as a multiple of the derivative of the denominator. Let $Q(x) = x^2 - 2x + 2 \Rightarrow Q'(x) = 2x-2$. To rewrite $x-5$, consider $x-5 = A(2x-2)+B$, splitting the integral into two pieces:

1. $\displaystyle\int \frac{Q'(x)}{Q(x)}\,dx$ (gives a log) 2. $\displaystyle\int \frac{dx}{Q(x)}$ (complete the square)

Now, clearly $A = \frac{1}{2}$ and $B = -4$, so $x-5 = \frac{1}{2}(2x-2) - 4$. Substitute this into integral

$$I = \int \frac{\frac{1}{2}(2x-2)-4}{x^2-2x+2}\,dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x+2}\,dx - 4\int \frac{dx}{x^2-2x+2}$$

We know $\displaystyle\int \frac{Q'(x)}{Q(x)}\,dx = \ln|Q(x)|$, and $(x-1)^2+1 = x^2-2x+2$.

Thus,

$$I = \frac{1}{2}\ln(x^2-2x+2) - 4\int \frac{dx}{(x-1)^2+1}$$

$$= \frac{1}{2}\ln(x^2-2x+2) - 4\arctan(x-1) + C.$$

1433. $\int \frac{x^3}{x^2+x+\frac{1}{2}}\,dx$

The first thing to notice is that the degree of x^3 is larger than the degree of $x^2+x+\frac{1}{2}$. This means we should first perform polynomial division. Dividing x^3 by $x^2+x+\frac{1}{2}$ we get

$$\frac{x^3}{x^2+x+\frac{1}{2}} = x-1+\frac{\frac{x}{2}+\frac{1}{2}}{x^2+x+\frac{1}{2}}$$

Therefore we have

$$\int \frac{x^3}{x^2+x+\frac{1}{2}}\,dx = \int (x-1)\,dx + \int \frac{\frac{x}{2}+\frac{1}{2}}{x^2+x+\frac{1}{2}}\,dx$$

The second part is like problem 1432, rewrite the numerator as a multiple of the derivative of the denominator, Let $Q(x)=x^2+x+\frac{1}{2}$ and $Q'(x)=2x+1$. Thus we write $\frac{x}{2}+\frac{1}{2} = A(2x+1)+B$, $A=\frac{1}{4}$ and $B=\frac{1}{4}$. The second integral then becomes

$$\int \frac{\frac{x}{2}+\frac{1}{2}}{x^2+x+\frac{1}{2}}\,dx = \frac{1}{4}\int \frac{2x+1}{x^2+x+\frac{1}{2}}\,dx + \frac{1}{4}\int \frac{dx}{x^2+x+\frac{1}{2}}$$

The first one is logarithmic as we know, the other we apply: $x^2+x+\frac{1}{2}=(x+\frac{1}{2})^2+\frac{1}{4}$, so

$$\int \frac{dx}{x^2+x+\frac{1}{2}} = \int \frac{dx}{(x+\frac{1}{2})^2+(\frac{1}{2})^2}$$ reducing it to

$1+t^2$, consider $t = 2x+1$, $dt = 2dx$, so

$$\int \frac{dx}{x^2+x+\frac{1}{2}} = \int \frac{\frac{1}{2}\,dt}{\frac{1}{4}(1+t^2)} = 2\int \frac{dt}{1+t^2}$$

$= 2\arctan(2x+1)$. Putting everything together,

$$\int \frac{x^3}{x^2+x+\frac{1}{2}}\,dx = \frac{x^2}{2} - x + \frac{1}{4}\ln\left(x^2+x+\frac{1}{2}\right) + \frac{1}{2}\arctan(2x+1) + C.$$

1433 combines all the ideas so far.

Problem 1434, $\int \frac{dx}{x(x^2+5)}$. This one is a rational function, whose denominator is already factored. Hence, natural is partial fraction decomposition.

$$\frac{1}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5} \Rightarrow 1 = A(x^2+5) + x(Bx+C)$$

$$\Longleftrightarrow 1 = Ax^2 + 5A + Bx^2 + Cx,\quad A = \frac{1}{5},\ B = -\frac{1}{5},\ C = 0.$$

So,

$$I = \frac{1}{5}\int \frac{dx}{x} - \frac{1}{5}\int \frac{x\,dx}{x^2+5}.$$

For the second one, $x\,dx = \frac{1}{2}\,d(x^2+5)$, so

$$\int \frac{dx}{x(x^2+5)} = \frac{1}{5}\ln|x| - \frac{1}{10}\ln(x^2+5) + C.$$

Problem 1435. $\displaystyle\int \frac{dx}{(x+2)^2(x+3)^2}$

This is again a rational function so we decompose it into partial fractions. When a denominator has repeated linear factors, the decomposition must include every power of each repeated factor up to its multiplicity. Therefore,

$$\frac{1}{(x+2)^2(x+3)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$$

Multiplying through,

$$1 = A(x+2)(x+3)^2 + B(x+3)^2 + C(x+3)(x+2)^2 + D(x+2)^2$$

set $x=-2$, yields $B=1$, set $x=-3$ yields $D=1$.
To find A & C, take $x=0$:

$$1 = A\cdot 2\cdot 9 + 9 + C\cdot 3\cdot 4 + 4 \iff 3A + 2C = -2$$

and take $x=-1$:

$$1 = A\cdot 4 + 4 + C\cdot 2 + 1 \iff 4A + 2C = -4$$

Putting these A-C eqs. together we get $A=-2$ and $C=2$.
Now Integrate!

$$I = -2\int \frac{dx}{x+2} + \int \frac{dx}{(x+2)^2} + 2\int \frac{dx}{x+3} + \int \frac{dx}{(x+3)^2}$$

Problem 1435. Evaluating each integral we get

$$I = -2\ln|x+2| - \frac{1}{x+2} + 2\ln|x+3| - \frac{1}{x+3} + C$$

$$= -2\ln\left|\frac{x+3}{x+2}\right| - \frac{1}{x+2} - \frac{1}{x+3} + C.$$

Problem 1436.

$$I = \int \frac{dx}{(x+1)^2(x^2+1)}$$

This again is a rational function, so we apply P.F.D.
The rule is: repeated linear factor gets a contribution one term for each power, and an irreducible quadratic gets a linear numerator.

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow 1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

For $x=-1$ we get $B=\frac{1}{2}$. Expanding the above

$$1 = A(x^3+x^2+x+1) + B(x^2+1) + Cx^3 + (2C+D)x^2 + D + (C+2D)x$$

$$= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D)$$

we get $A+C=0$, $A+B+2C+D=0$, $A+C+2D=0$ and $A+B+D=1$. We see that $A=-C$, so $D=0$. Since $B=1/2$, $A+\frac{1}{2}=1 \Rightarrow A=\frac{1}{2}$ and $C=-\frac{1}{2}$. Thus,

$$I=\frac{1}{2}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{(x+1)^2}-\frac{1}{2}\int\frac{x\,dx}{x^2+1}$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{2}\cdot\frac{1}{(x+1)}-\frac{1}{4}\ln(x^2+1)+C$$

$$=\frac{1}{4}\ln\left(\frac{(x+1)^2}{x^2+1}\right)-\frac{1}{2(x+1)}+C.$$

Problem 1437.

$$I=\int\frac{dx}{(x^2+2)^2}.$$

Here, P.F.D. reproduces the same integral so we need to know some other method. In 1431 we learned that expressions involving x^2+a^2 suggest the arctangent, but x^2+2 is squared so it is not standard. The idea is to turn x^2+2 into something simple. E.g., since $1+\tan^2 t=\sec^2 t$ we choose $x=\sqrt{2}\tan t$. Then $dx=\sqrt{2}\sec^2 t\,dt$, and we have $(x^2+2)^2=(2\tan^2 t+2)^2=[2(\tan^2 t+1)]^2=4\sec^4 t$.

So, $$I=\int\frac{\sqrt{2}\sec^2 t\,dt}{4\sec^4 t}=\frac{\sqrt{2}}{4}\int\cos^2 t\,dt$$

$$=\frac{\sqrt{2}}{4}\int\frac{1}{2}(1+\cos 2t)\,dt=\frac{\sqrt{2}}{8}\left(t+\frac{1}{2}\sin 2t\right)+C$$

we have $I = \frac{\sqrt{2}}{8} t + \frac{\sqrt{2}}{16} \sin 2t + C$ and $x = \sqrt{2} \tan t$.

So $\tan t = \frac{x}{\sqrt{2}}$, hence use $\sin 2t = 2 \sin t \cos t$

where $2 \sin t \cos t = \frac{2 \sin t \cos t}{\sin^2 t + \cos^2 t}$, divide by $\cos^2 t$:

$\sin 2t = \frac{2 \tan t}{1 + \tan^2 t}$. Therefore $\sin 2t = \frac{2(x/\sqrt{2})}{1 + x^2/2}$

$= \frac{2\sqrt{2}\, x}{(x^2+2)}$. Substituting into I:

$$I = \frac{\sqrt{2}}{8} \arctan \frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{16} \cdot \frac{2\sqrt{2}\, x}{x^2+2} + C$$

$$= \frac{1}{4\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + \frac{1}{4} \cdot \frac{1}{(x^2+2)} + C$$

Problem 1438.

$$\int \frac{dx}{x^4 - 2x^2 + 1}$$

The denominator is a perfect square; $(x^4 - 2x^2 + 1) = (x^2-1)^2$. And $(x^2-1)^2 = (x+1)^2(x-1)^2$, i.e., the denominator factors into repeated linear factors. So,

$$\frac{1}{(x-1)^2(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$$

Problem 1438. Multiplying through,

$$1 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$

Set $x=1$, $B=\frac{1}{4}$. Set $x=-1$, then $D=\frac{1}{4}$. For $x=0$ we have

$$1 = A(-1)(1)^2 + \tfrac{1}{4}(1) + C(-1)^2(1) + \tfrac{1}{4}(-1)^2$$

$$\Longleftrightarrow$$

$$1 = -A + \tfrac{1}{4} + C + \tfrac{1}{4}$$

$$\Longleftrightarrow$$

$$C - A = \tfrac{1}{2}$$

similarly for $x=2$, $1 = A(1)(3)^2 + \frac{1}{4}(3)^2 + C(1)(3) + \frac{1}{4}(1)$

$\Longleftrightarrow$ $1 = 9A + \frac{9}{4} + 3C + \frac{1}{4}$

$\Longleftrightarrow$ $3A + C = -\frac{1}{2}$, which gives $A = -\frac{1}{4}$ and $C = \frac{1}{4}$

Thus the integral becomes,

$$I = \int \frac{dx}{x^4 - 2x + 1} = -\frac{1}{4}\int \frac{dx}{(x-1)} + \frac{1}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{4}\int \frac{dx}{(x+1)} + \frac{1}{4}\int \frac{dx}{(x+1)^2}$$

$$= -\frac{1}{4}\ln|x-1| - \frac{1}{4}\frac{1}{(x-1)} + \frac{1}{4}\ln|x+1| - \frac{1}{4}\frac{1}{(x+1)}$$

$$= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C,$$

Problem 1439.

$$I = \int \frac{x\,dx}{(x^2-x+1)^3}$$

The denominator $Q(x) = x^2 - x + 1$ is a quadratic and the numerator is linear. This suggests that we should rewrite the numerator as a multiple of the derivative of the denominator plus a constant (just like in 1432).

Since $Q'(x) = 2x - 1$ we have $x = A(2x-1) + B$ where $A = B = 1/2$. The integral splits into

$$I = \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)^3}\,dx + \frac{1}{2}\int \frac{dx}{(x^2-x+1)^3}$$

The first term becomes $-\dfrac{1}{4(x^2-x+1)^2}$ but the second term needs some help. Complete the square s.t. $x^2-x+1 = (x-\frac{1}{2})^2 + \frac{3}{4}$. Moreover, let $y = x - \frac{1}{2}$, then the denominator has the form $(y^2+a^2)^3$ with $a = \frac{\sqrt{3}}{2}$.

Naturally this suggests the tangent sub: $y = \frac{\sqrt{3}}{2}\tan t$ with $dy = \frac{\sqrt{3}}{2}\sec^2 t\,dt$. This then becomes,

$$y^2 + a^2 = \frac{3}{4}\tan^2 t + \frac{3}{4} = \frac{3}{4}(1+\tan^2 t) = \frac{3}{4}\sec^2 t.$$

Problem 1439. Therefore,

$$\int \frac{dy}{(y^2+\frac{3}{4})^3} = \int \frac{\frac{\sqrt{3}}{2}\sec^2 t\, dt}{(\frac{3}{4}\sec^2 t)^3} = \frac{\sqrt{3}}{2}\cdot\frac{64}{27}\int \cos^4 t\, dt$$

$$= \frac{32\sqrt{3}}{27}\int \cos^4 t\, dt.$$ Now, recall that

$$\cos^2 t = \frac{1+\cos 2t}{2} \quad \text{so} \quad \cos^4 t = \frac{1}{4}(1+2\cos 2t+\cos^2 2t)$$

$$= \frac{1}{4}\left[1+2\cos 2t+\frac{1+\cos 4t}{2}\right]$$

$$= \frac{1}{8}\left[3+4\cos 2t+\cos 4t\right].$$

Hence,

$$\frac{32\sqrt{3}}{27}\int \cos^4 t\, dt = \frac{32\sqrt{3}}{27}\int \frac{1}{8}\left[3+4\cos 2t+\cos 4t\right] dt$$

$$= \frac{4\sqrt{3}}{27}\left(3t+4\cdot\frac{\sin 2t}{2}+\frac{\sin 4t}{4}\right)$$

$$= \frac{4\sqrt{3}}{9}t+\frac{8\sqrt{3}}{27}\sin 2t+\frac{\sqrt{3}}{27}\sin 4t$$

From our substitution, we find $t = \arctan\left(\frac{2x-1}{\sqrt{3}}\right)$.

For $\sin 2t = 2\sin t\cos t$, divide by $\cos^2 t$:

$$\sin 2t = \frac{2\tan t}{1+\tan^2 t} = \frac{2(2y/\sqrt{3})}{1+4y^2/3} = \frac{2\sqrt{3}(2x-1)}{3+(2x-1)^2}$$

Problem 1439.

That is, $\sin 2t = \dfrac{\sqrt{3}(2x-1)}{2(x^2-x+1)}$ and therefore,

$$\frac{8\sqrt{3}}{27}\sin 2t = \frac{8\sqrt{3}}{27}\cdot\frac{\sqrt{3}}{2}\,\frac{(2x-1)}{(x^2-x+1)} = \frac{4(2x-1)}{9(x^2-x+1)}$$

For $\sin 4t = 2\sin 2t\cos 2t$, where $\cos 2t = \dfrac{1-\tan^2 t}{1+\tan^2 t}$

s.t., $\cos 2t = \dfrac{1-\frac{(2x-1)^2}{3}}{1+\frac{(2x-1)^2}{3}} = \dfrac{3-(2x-1)^2}{3+(2x-1)^2}$ where

$3+(2x-1)^2 = 4(x^2-x+1)$, so $\cos 2t = \dfrac{3-(2x-1)^2}{4(x^2-x+1)}$

therefore $\sin 4t = 2\left[\dfrac{\sqrt{3}(2x-1)}{2(x^2-x+1)}\right]\left[\dfrac{3-(2x-1)^2}{4(x^2-x+1)}\right]$

Hence $\sin 4t = \dfrac{\sqrt{3}(2x-1)(3-(2x-1)^2)}{4(x^2-x+1)^2}$

The entire integral is,

$$\int\frac{dx}{(x^2-x+1)^3} = \frac{4}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + \frac{4(2x-1)}{9(x^2-x+1)} + \frac{(2x-1)(3-(2x-1)^2)}{36(x^2-x+1)^2}$$

$$+\ C$$

Problem 1440.

$$I = \int \frac{3-4x}{(1-2\sqrt{x})^2}\,dx$$

The important feature is $\sqrt{x}$. The substitution principle is to choose a new variable that removes the radical and makes the integrand rational.

You could set $\sqrt{x} = t$, but here we have something better, i.e., $u = 1 - 2\sqrt{x}$, or $(\sqrt{x})^2 = (1-u)^2/4$.
The differential is $dx = -\frac{1-u}{2}\,du$. The numerator becomes

$3 - 4x = 3 - 4(1-u)^2/4 = 3 - (1-u)^2$. The integral is

$$I = -\int \frac{u^2+2u+2}{u^2} \cdot \frac{1-u}{2}\,du$$

$$= -\frac{1}{2}\int \frac{u^3 - 3u^2 + 2}{u^2}\,du$$

$$= -\frac{1}{2}\int \left(u - 3 + \frac{2}{u^2}\right) du$$

$$= -\frac{1}{2}\left(\frac{u^2}{2} - 3u - \frac{2}{u}\right) = -\frac{1}{4}u^2 + \frac{3}{2}u + \frac{1}{u} + C$$

which is with $u = 1 - 2\sqrt{x}$:

$$I = -\frac{1}{4}(1-2\sqrt{x})^2 + \frac{3}{2}(1-2\sqrt{x}) + \frac{1}{1-2\sqrt{x}} + C.$$

Problem 1441.

$$I = \int \frac{(\sqrt{x}+1)^2}{x^3}\,dx$$

There is no need for excessive methods if we first can simplify the algebra.

$$(\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1$$

So, $$\frac{(\sqrt{x}+1)^2}{x^3} = \frac{x+2\sqrt{x}+1}{x^3} = \frac{1}{x^2} + \frac{2\sqrt{x}}{x^5} + \frac{1}{x^3}$$

$$I = \int \left(x^{-2} + 2x^{-5/2} + x^{-3}\right)dx$$

$$= -\frac{1}{x} - \frac{4}{3x^{3/2}} - \frac{1}{2x^2} + C$$

If the only complication is the radical in the numerator see if you can simplify first such that everything becomes ordinary powers.

Problem 1442.

$$I = \int \frac{dx}{\sqrt{x^2+x+1}}$$

The denominator is a quadratic trinomial. The method for this class of problems is to complete the square first, then use ~~trigonometric or hyperbolic substitution~~ standard tabular integrals.

Problem 1442. Completing the square,

$$x^2 + x + 1 = (x + \tfrac{1}{2})^2 + \tfrac{3}{4}. \text{ Thus,}$$

$$I = \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}$$

which is of the form $\sqrt{u^2 + a^2}$, so set $u = x + \frac{1}{2}$ with $du = dx$:

$$I = \int \frac{du}{\sqrt{u^2 + \frac{3}{4}}} = \ln | u + \sqrt{u^2 + \tfrac{3}{4}} \, | + C$$

i.e., $I = \ln | x + \frac{1}{2} + \sqrt{x^2 + x + 1} \, | + C.$

Problem 1443.

$$\int \frac{1 - \sqrt[3]{2x}}{\sqrt{2x}} \, dx$$

Idea is to simplify the irrational expression into ordinary powers.

Note that $\dfrac{1 - \sqrt[3]{2x}}{\sqrt{2x}} = (2x)^{-1/2} - (2x)^{-1/6}$

so $I = \int (2x)^{-1/2} \, dx - \int (2x)^{-1/6} \, dx$

Problem 1443. We had

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x}}\, dx - \frac{1}{\sqrt[6]{2}} \int x^{-1/6}\, dx$$

$$= \frac{1}{\sqrt{2}} \cdot 2\sqrt{x} - \frac{1}{\sqrt[6]{2}} \cdot \frac{6}{5} x^{5/6} + C$$

$$= \sqrt{2x} - \frac{3}{5}(2x)^{5/6} + C.$$

Problem 1444.

$$\int \frac{dx}{(\sqrt[3]{x^2} + \sqrt[3]{x})^2}$$

Rational expressions with fractional powers are ripe for a certain type of substitution. This consists of choosing a new variable whose power is the least common multiple of the root indices. Here we have only cube roots $x^{1/3}$ and $x^{2/3}$, so $x = z^3$. This is the method, i.e., replace x by a cube s.t. every cube root becomes an ordinary integer power.

$$I = \int \frac{dx}{(\sqrt[3]{x^2} + \sqrt[3]{x})^2} = \int \frac{3z^2\, dz}{(z^2 + z)^2}$$

Factor the denominator $(z^2 + z)^2 = z^2(z+1)^2$, Problem 1444. So,

$$I = \int \frac{3z^2\,dz}{z^2(z+1)^2} = \int \frac{3\,dz}{(z+1)^2}$$

$$= 3\int \frac{dz}{(z+1)^2} = -\frac{3}{z+1} + C$$

and since $z = \sqrt[3]{x}$,

$$I = -\frac{3}{\sqrt[3]{x}+1} + C$$

(Takeaway: make irrational powers rational at once.)

Problem 1445.

$$I = \int \frac{2x+1}{\sqrt{(4x^2-2x+1)^3}}\,dx$$

The denominator is a quadratic trinomial and the numerator is linear. This suggests we should try to make the numerator a multiple of the derivative of the denominator plus a constant.

$$Q(x) = 4x^2 - 2x + 1 \Rightarrow Q'(x) = 8x - 2.$$

Then we can substitute $t = 8x-2$, $x = \frac{t+2}{8}$ and $dx = dt/8$.

Problem 1445. The quadratic becomes,

$$4x^2 - 2x + 1 = 4\left(\frac{t+2}{8}\right)^2 - 2\,\frac{t+2}{8} + 1 = \frac{t^2+12}{16}.$$

The numerator is $2x+1 = 2\,\frac{t+2}{8} + 1 = \frac{t+6}{4}$.

Therefore,

$$I = \int \frac{\frac{t+6}{4} \cdot \frac{dt}{8}}{\left(\frac{t^2+12}{16}\right)^{3/2}} = 2\int \frac{t+6}{(t^2+12)^{3/2}}\,dt$$

Consider $\sqrt{t^2+12} = \sqrt{t^2+(2\sqrt{3})^2}$. If an integral contains the radical $\sqrt{x^2+a^2}$, set $x = a\tan u$.
So, let's set $t = 2\sqrt{3}\tan\theta$ & $dt = 2\sqrt{3}\sec^2\theta\,d\theta$.
Furthermore, $t^2+12 = 12\tan^2\theta + 12 = 12\sec^2\theta$
The whole integral becomes,

$$I = 2\int \frac{(2\sqrt{3}\tan\theta + 6)(2\sqrt{3}\sec^2\theta)}{(12\sec^2\theta)^{3/2}}\,d\theta$$

$$= 2\int \frac{4\cdot 3\tan\theta\sec^2\theta + 2\sqrt{3}\cdot 6\sec^2\theta}{24\sqrt{3}\sec^3\theta}\,d\theta$$

$$= \int \frac{2\sqrt{3}\tan\theta + 6}{6}\cos\theta\,d\theta$$

$$= \int\left(\frac{1}{\sqrt{3}}\sin\theta + \cos\theta\right)d\theta$$

Problem 1445. The integral becomes,

$$I = \frac{1}{\sqrt{3}} \cdot (-1) \cos\theta + \sin\theta . \quad \text{Since } \tan\theta = \frac{t}{2\sqrt{3}},$$

we have $\sin\theta = \dfrac{t}{\sqrt{t^2+12}}$ & $\cos\theta = \dfrac{2\sqrt{3}}{\sqrt{t^2+12}}$.

Thus,

$$I = -\frac{1}{\sqrt{3}} \frac{2\sqrt{3}}{\sqrt{t^2+12}} + \frac{t}{\sqrt{t^2+12}} + C$$

and with $t = 8x - 2$ we get

$$I = -\frac{2}{\sqrt{(8x-2)^2+12}} + \frac{8x-2}{\sqrt{(8x-2)^2+12}} + C$$

$$= -\frac{2}{\sqrt{64x^2-32x+16}} + \frac{8x-2}{\sqrt{64x-32x+16}} + C$$

$$= \frac{2x-1}{\sqrt{4x^2-2x+1}} + C$$

Problem 1446.

$$I = \int \frac{dx}{\sqrt[4]{5-x} + \sqrt{5-x}}$$

The idea is to choose a substitution that turns all the fractional powers into integer powers. The exponents are $1/4$ and $1/2$, choosing a fourth root is sufficient. We set $t = \sqrt[4]{5-x} \Rightarrow x = 5 - t^4$ with $dx = -4t^3\,dt$. Subbing into I:

$$I = -\int \frac{4t^3}{t+t^2}\,dt = -4\int \frac{t^2}{1+t}\,dt$$

The irrational has become rational! Now $\deg(t^2) > \deg(t+1)$, so Demidovich says "take out the whole part first," i.e., do polynomial division.

$$\begin{array}{l|l} \multicolumn{1}{l}{t-1} & \\ \hline t^2 & t+1 \\ -(t^2+t) & \\ \hline -t & \\ -(-t)+1 & \\ \hline 1 & \end{array}$$

So $t^2 = (t+1)(t-1) + 1$, thus the integral is

$$I = -4\int \left(t - 1 + \frac{1}{t+1}\right) dt$$

Problem 1446. The integral becomes

$$I = -4\left[\frac{t^2}{2} - t + \ln|x-1|\right] + C$$

$$= -2t^2 + 4t - 4\ln|t+1| + C.$$

with $t = \sqrt[4]{5-x}$, this becomes,

$$I = -2\sqrt{5-x} + 4\sqrt[4]{5-x} - 4\ln|\sqrt[4]{5-x} + 1| + C$$

Problem 1447.

$$I = \int \frac{x^2}{\sqrt{(x^2-1)^3}}\,dx$$

Demidovich tells us that for $\sqrt{x^2-a^2}$ forms, the hyperbolic substitution $x = a\cosh t$ is possible.

Since we have $\sqrt{x^2-1}$, then $a=1$ and $x = \cosh t$. The reason this works is because of $\cosh^2 t - 1 = \sinh^2 t$. Also we have $dx = \sinh t\, dt$

The whole integral becomes,

$$I = \int \frac{\cosh^2 t \sinh t\, dt}{(\sinh^2 t)^{3/2}} = \int \frac{\cosh^2 t}{\sinh^2 t}\,dt$$

Problem 1447. The integral becomes

$$I = \int \frac{\cosh^2 t}{\sinh^2 t}\,dt \cancel{= \int \coth^2 t \, dt}$$

~~Since~~ $\coth^2 t = 1 + \operatorname{csch}^2 t$, ~~then~~

$$\cancel{I = \int (1 + \operatorname{csch}^2 t)\,dt = t - \coth t + C}$$

Here we might use the identity $\cosh^2 t = 1 + \sinh^2 t$,

$$I = \int dt + \int \frac{dt}{\sinh^2 t}.$$

Now we simply differentiate $\left(\frac{\cosh t}{\sinh t}\right)$ by the quotient rule,

$$\frac{d}{dt}\left(\frac{\cosh t}{\sinh t}\right) = \frac{\sinh^2 t - \cosh^2 t}{\sinh^2 t} = -\frac{1}{\sinh^2 t}.$$

Therefore,

$$I = \int dt + \int \frac{dt}{\sinh^2 t} = t - \frac{\cosh t}{\sinh t} + C$$

with $x = \cosh t$ we have

$\sinh^2 t = \cosh^2 t - 1 = x^2 - 1$. To find t, consider

$\cosh t = \frac{e^t + e^{-t}}{2} \Rightarrow 2x = e^t + e^{-t}$, multiply with e^t:

$2xe^t = e^{2t} + 1 \Rightarrow$ quadratic eq. in e^t

$$\Rightarrow e^t = \frac{2x \pm \sqrt{(2x)^2 - 4}}{2}$$

Problem 1447. Take $e^t = x + \sqrt{x^2 - 1}$, and then

$\ln(e^t) = \ln(x + \sqrt{x^2 - 1})$. Finally,

$$I = \ln\left|x + \sqrt{x^2-1}\right| - \frac{x}{\sqrt{x^2-1}} + C.$$

Problem 1448.

$$I = \int \frac{x\,dx}{(1+x^2)\sqrt{1-x^4}}$$

The fact of the $x\,dx$ in the numerator suggests $t = x^2$. The integral becomes

$$I = \frac{1}{2}\int \frac{dt}{(1+t)\sqrt{1-t^2}}$$

For a radical of the form $\sqrt{a^2 - t^2}$, the Demidovich sub-sandwich is $t = a \sin u$. Naturally, $dt = \cos u\, du$ and $\sqrt{1-t^2} = \cos u$, so,

$$I = \frac{1}{2}\int \frac{\cos u\, du}{(1+\sin u)\cos u} = \frac{1}{2}\int \frac{du}{1+\sin u}$$

Here, we need some further manipulation....

Problem 1448. If we multiply both numerator & denominator with $1-\sin u$:

$$I = \frac{1}{2}\int \left(\frac{1-\sin u}{\cos^2 u}\right) du$$

$$= \frac{1}{2}\int \frac{1}{\cos^2 u}\, du - \frac{1}{2}\int \frac{\sin u}{\cos^2 u}\, du$$

where the first piece is $\frac{1}{2}\tan u$ and for the second piece, set $\Lambda = \cos u$, $d\Lambda = -\sin u\, du$, then $\int \frac{\sin u}{\cos^2 u}\, du = -\int \frac{d\Lambda}{\Lambda^2} = \frac{1}{\Lambda} = \frac{1}{\cos u}$, thus

$I = \frac{1}{2}\left[\tan u - \frac{1}{\cos u}\right] + C$. Remember $\sin u = t$ so $\cos u = \sqrt{1-t^2}$ and $\tan u = \frac{t}{\sqrt{1-t^2}}$. Therefore,

$$I = \frac{1}{2}\left[\frac{t}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-t^2}}\right] + C = \frac{t-1}{2\sqrt{1-t^2}} + C$$

and since $t = x^2$ we get

$$I = \frac{x^2-1}{2\sqrt{1-x^4}} + C$$

or, $I = -\frac{\sqrt{1-x^4}}{2(1+x^2)} + C$.

Problem 1449.

$$I = \int \frac{x\,dx}{\sqrt{1-2x^2-x^4}}$$

The numerator contains $x\,dx$ and the denominator has x^2, x^4, which naturally suggests $t = x^2$, $dt = 2x\,dx$. Then,

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{1-2t-t^2}}$$

Now we have the quadratic trinomial, so let's transform it to a sum or difference of squares. For a form $\sqrt{m^2 - z^2}$, Demidovich gives $z = m\sin u$.

Completing the square $1-2t-t^2 = -(t+1)^2 + 2$.

Therefore,

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{2-(t+1)^2}}$$

with $m = \sqrt{2}$ and $z = t+1$, thus we set $t+1 = \sqrt{2}\sin u$ with $dt = \sqrt{2}\cos u\,du$. Then,

$$I = \frac{1}{2}\int \frac{\sqrt{2}\cos u\,du}{\sqrt{2-2\sin^2 u}}$$

$$= \frac{1}{2}\int du = \frac{u}{2} + C. \text{ So,}$$

$$I = \frac{1}{2}\arcsin\left(\frac{x^2+1}{\sqrt{2}}\right) + C.$$

Problem 1450.

$$I = \int \frac{x+1}{(x^2+1)^{3/2}}\,dx$$

So again, when the numerator is linear and contains part of the derivative of the quadratic polynomial, you should take that out of the numerator.

Thus, $Q(x) = x^2+1 \Rightarrow Q'(x) = 2x$.

And $x+1 = A(2x)+B$, $A = \frac{1}{2}$, $B = 1$.
The integral becomes

$$I = \frac{1}{2}\int \frac{2x\,dx}{(x^2+1)^{3/2}} + \int \frac{dx}{(x^2+1)^{3/2}}$$

For the first integral, $u = x^2+1$, $du = 2x\,dx$:

$$\frac{1}{2}\int \frac{2x\,dx}{(x^2+1)^{3/2}} = \frac{1}{2}\int u^{-3/2}\,du = -\frac{1}{\sqrt{u}}.$$

The $\int \frac{dx}{(x^2+1)^{3/2}}$ we can handle by $x = \tan u$.

Demidovich gives this sub-sandwich for expressions on the form $\sqrt{x^2+a^2}$.

Problem 1450. Since $x = \tan u$, $dx = \sec^2 u \, du$
and $x^2 + 1 = \tan^2 u + 1 = \sec^2 u$.

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{\sec^2 u \, du}{(\sec^2 u)^{3/2}} = \int \cos u \, du = \sin u$$

where $\sin u = \dfrac{x}{\sqrt{x^2+1}}$ because $x = \tan u$

combining all this,

$$I = -\frac{1}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}} + C$$

$$= \frac{x-1}{\sqrt{x^2+1}} + C.$$