1.10 Problem 10

Two spin-1/2 particles with magnetic moment $\hat{\vec{M}} = g\hat{\vec{S}}$ are placed at some distance. The particles interact as magnetic dipoles. What is the most favorable spin configuration of the particles?

The notion of "most favorable," means that we want to find the configuration with the lowest energy. Typically, we can think about a spin-1/2 particle as a tiny bar magnet, at least here anyway. A bar magnet has two poles, and the question then is really about how to align two of these magnets such that their interaction energy is minimized. If you already have some intuition classically, you can infer that the result must be a tail to head alignment.

The interaction energy of the dipoles can be seen as one dipole in the field of the other dipole's magnetic field:

$$V = -\boldsymbol{M}_2 \cdot \boldsymbol{B}_1 \tag{1}$$

The magnetic field of a dipole is

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{M}\cdot\hat{r})\hat{r} - \boldsymbol{M}] \tag{2}$$

so

$$V = -\boldsymbol{M}_2 \cdot \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{M}_1\cdot\hat{r})\hat{r} - \boldsymbol{M}_1] \tag{3}$$

$$= \frac{\mu_0}{4\pi r^3}[\boldsymbol{M}_1\cdot\boldsymbol{M}_2 - 3(\boldsymbol{M}_1\cdot\hat{r})(\boldsymbol{M}_2\cdot\hat{r})] \tag{4}$$

The dipoles are located at a distance r from each other, suppose that $\hat{r} = \hat{z}$, then the interaction energy is

$$\begin{aligned} V &= \frac{\mu_0}{4\pi r^3}[g\hat{S}_1\cdot g\hat{S}_2 - 3(g\hat{S}_1\cdot\hat{z})(g\hat{S}_2\cdot\hat{z})] \\ &= \frac{\mu_0 g^2}{4\pi r^3}[S_{1x}S_{2x} + S_{1y}S_{2y} - 2S_{1z}S_{2z}] \end{aligned} \tag{5}$$

Using the fact that $S_\pm = S_x \pm iS_y$, then $S_{1x} = \frac{1}{2}(S_{1+}+S_{1-})$, $S_{2x} = \frac{1}{2}(S_{2+}+S_{2-})$, $S_{1y} = \frac{1}{2i}(S_{1+} - S_{1-})$ and $S_{2y} = \frac{1}{2i}(S_{2+} - S_{2-})$.

Simplifying the x and y-components in the bracket of Eq. (5):

$$\begin{aligned} S_{1x}S_{2x} + S_{1y}S_{2y} &= \frac{1}{4}(S_{1+} + S_{1-})(S_{2+} + S_{2-}) + \frac{1}{4i^2}(S_{1+} - S_{1-})(S_{2+} - S_{2-}) \\ &= \frac{1}{4}(S_{1+}S_{2+} + S_{1+}S_{2-} + S_{1-}S_{2+} + S_{1-}S_{2-} - S_{1+}S_{2+} + S_{1+}S_{2-} + S_{1-}S_{2+} - S_{1-}S_{2-}) \\ &= \frac{1}{4}(2S_{1+}S_{2-} + 2S_{1-}S_{2+}) \end{aligned} \tag{6}$$

$$V = \frac{\mu_0 g^2}{4\pi r^3}[\frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}) - 2S_{1z}S_{2z}] \tag{7}$$

The basis for the spin-1/2 particles is $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$. This is now an eigenvalue problem $H|\psi\rangle = E|\psi\rangle$. Before acting on the different states with $\hat{H}$, let $C = \frac{\mu_0 g^2}{4\pi r^3}$. For the state $|\uparrow\uparrow\rangle$, we have $2S_{1z}S_{2z}|\uparrow\uparrow\rangle = \frac{\hbar^2}{2}|\uparrow\uparrow\rangle$ and $\frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+})|\uparrow\uparrow\rangle = 0$, giving $\hat{H}|\uparrow\uparrow\rangle = -\frac{C\hbar}{2}|\uparrow\uparrow\rangle$. Similarly for $|\downarrow\downarrow\rangle$ we have $\hat{H}|\downarrow\downarrow\rangle = -\frac{C\hbar}{2}|\downarrow\downarrow\rangle$.

For the mixed state $|\uparrow\downarrow\rangle$ we have $2S_{1z}S_{2z}|\uparrow\downarrow\rangle = -\frac{\hbar^2}{2}|\uparrow\downarrow\rangle$ and $\frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+})|\uparrow\downarrow\rangle = \frac{\hbar^2}{2}|\downarrow\uparrow\rangle$ resulting in $\hat{H}|\downarrow\uparrow\rangle = C[\frac{\hbar^2}{2}|\downarrow\uparrow\rangle + \frac{\hbar^2}{2}|\uparrow\downarrow\rangle]$. Likewise for $|\downarrow\uparrow\rangle$ we have $2S_{1z}S_{2z}|\downarrow\uparrow\rangle = -\frac{\hbar^2}{2}|\downarrow\uparrow\rangle$ and $\frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+})|\downarrow\uparrow\rangle = \frac{\hbar^2}{2}|\uparrow\downarrow\rangle$ yielding $\hat{H}|\downarrow\uparrow\rangle = C[\frac{\hbar^2}{2}|\uparrow\downarrow\rangle - \frac{\hbar^2}{2}|\downarrow\uparrow\rangle]$. In this mixed space, the matrix for the Hamiltonian is

$$\frac{C\hbar^2}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{8}$$

with eigenvectors $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and $\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$ with eigenvalues $E = C\hbar^2$ and $E = 0$ respectively. The lowest energy states are therefore $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$.

Answer : The most favorable spin configuration is either $|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$