1.12 Problem 124

Show, that quasiclassical solution gives $E = \hbar\omega(n + 1/2)$ *for the oscillator potential.*

By quasiclassical we mean a system that is still quantum, but close enough to classical motion that classical quantities can be use. The oscillator potential is $\frac{1}{2}m\omega^2 x^2$. In QM, the state is described by a wavefunction, while in classical mechanics a particle is described by a path determined by the equations of motion. Thus, we treat the classical picture as a limiting case of the quantum one.

The quasiclassical quantization rule for a 1D system between two smooth turning points is

$$\int_{x_1}^{x_2} p(x)dx = \pi\hbar(n + \frac{1}{2}) \tag{24}$$

where $p(x) = \sqrt{2m(E - U(x))}$ and the turning points are determined by $E = U(x)$. For the latter, $E = \frac{1}{2}m\omega^2 x^2 \implies x = \pm\sqrt{\frac{2E}{m\omega^2}} := \pm a$. Rewriting the momentum to

$$p(x) = \sqrt{2m(\frac{1}{2}m\omega^2 a^2 - \frac{1}{2}m\omega^2 x^2)} = 2m\sqrt{a^2 - x^2} \tag{25}$$

so

$$\int_{-a}^{+a} p(x)\,dx = m\omega \int_{-a}^{+a} \sqrt{a^2 - x^2}\,dx = m\omega\frac{\pi a^2}{2} = \frac{\pi E}{\omega} \tag{26}$$

The quantization condition becomes

$$\frac{\pi E}{\omega} = \pi\hbar(n + \frac{1}{2}) \longleftrightarrow E_n = \hbar\omega(n + \frac{1}{2}) \tag{27}$$

which is the standard result.