1.11 Problem 13

Two neutrons are in the harmonic potential $U = \frac{\alpha r^2}{2}$. Their spins are interacting according to $\lambda\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$. Find what conditions on λ and α must be satisfied so that the total spin in the ground state is 1.

For $S = 1$ to be the ground state then the energy associated with the triplet state must be smaller than the singlet state, i.e., $E_{triplet} < E_{singlet}$. The energies associated with each state is a combination of the harmonic potential and the spin. For a harmonic potential in 3D, the energy levels are $E_n = \hbar\omega(N + 3/2)$ for $N = 0, 1, 2, ...$ etc. The interaction Hamiltonian is $H_{int} = \lambda\hat{S}_1 \cdot \hat{S}_2$, where

$$\hat{S}_1 \cdot \hat{S}_2 = \frac{1}{2}(\boldsymbol{S}^2 - \hat{S}_1^2 - \hat{S}_2^2) \tag{17}$$

With each spin operator taking eigenvalues $\boldsymbol{S}_i^2 = s(s+1)\hbar^2$. In the case of the singlet state ($S = 0$) we have

$$\frac{1}{2}(\boldsymbol{S}^2 - \hat{S}_1^2 - \hat{S}_2^2) = \frac{1}{2}(0 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2) = -\frac{3}{4}\hbar^2 \tag{18}$$

and for the triplet state ($S = 1$) we instead get

$$\frac{1}{2}(\boldsymbol{S}^2 - \hat{S}_1^2 - \hat{S}_2^2) = \frac{1}{2}(2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2) = \frac{1}{4}\hbar^2 \tag{19}$$

Now, neutrons are fermions, so for $S = 0$ (antisymmetric) its spatial wave function needs to be symmetric, i.e., both neutrons can be in the ground state $N = 0$. This means that the total energy is

$$E_{singlet} = 2 * \hbar\omega(0 + 3/2) - \frac{3}{4}\lambda\hbar^2 = 3\hbar\omega - \frac{3}{4}\lambda\hbar^2 \tag{20}$$

The triplet state, $S = 1$, is symmetric, which means that the spatial wave functions needs to be antisymmetric, i.e., one neutron can be in the ground state but the other has to be in an excited state. This means that the total energy is

$$E_{triplet} = \frac{3}{2}\hbar\omega + \frac{5}{2}\hbar\omega + \frac{1}{4}\lambda\hbar^2 = 4\hbar\omega + \frac{1}{4}\lambda\hbar^2 \tag{21}$$

Using the condition given in the text, $E_{triplet} < E_{singlet}$:

$$4\hbar\omega + \frac{1}{4}\lambda\hbar^2 < 3\hbar\omega - \frac{3}{4}\lambda\hbar^2 \implies \lambda < -\frac{\omega}{\hbar} = -\frac{1}{\hbar}\sqrt{\frac{\alpha}{m}} \tag{22}$$

For this to hold, $\alpha > 0$. (*We have used $\alpha = m\omega^2$ as deduced from the harmonic potential.*)