1.9 Problem 9

For a hydrogen atom find the transition probability from the ground state to the first excited state after a weak electric field is suddenly turned on.

We interpret "suddenly turned on," as if the hydrogen does not have time to react to the e-field such that it stays in the 1s ground state. The perturbation operator is $\hat{V} = e\mathcal{E}z$ (see §77 Vol 3) where e is the charge of the electron, $\mathcal{E}$ is the magnitude of the electric field and with z as the axis in direction of the field.

The corresponding Hamiltonian is

$$H = H_0 + V \tag{1}$$

The formula for perturbations independent of time is

$$\psi_n^{(1)} = \sum_{m \neq n} \frac{V_{mn}}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)} \tag{2}$$

where $V_{mn} = \langle m|V|n\rangle$. The new eigenstate is the old one, plus a little bit of every other state, i.e.,

$$|\tilde{n}\rangle = |n\rangle + \sum_{m \neq n} \frac{\langle m|V|n\rangle}{E_n^{(0)} - E_m^{(0)}} |m\rangle \tag{3}$$

The probability of finding the atom in the excited state $|\tilde{n}\rangle$ given that it is in the 1s state, is $|\langle\tilde{n}|1s\rangle|^2$. Plugging in the formula for $|\tilde{n}\rangle$

$$\langle\tilde{n}|1s\rangle = \langle n|1s\rangle + \sum_{m \neq n} \frac{\langle n|V|m\rangle}{E_n^{(0)} - E_m^{(0)}} \langle m|1s\rangle \tag{4}$$

Here $|n\rangle$ is an excited state, so $\langle n|1s\rangle = 0$, and only terms with $m = 1s$ survive:

$$\langle\tilde{n}|1s\rangle = \frac{\langle n|V|1s\rangle}{E_n^{(0)} - E_{1s}^{(0)}} \tag{5}$$

The probability of finding the system in state $|\tilde{n}\rangle$ is

$$|\langle\tilde{n}|1s\rangle|^2 = |\frac{\langle n|V|1s\rangle}{E_n^{(0)} - E_{1s}^{(0)}}|^2 \tag{6}$$

Now, $\langle n|V|1s\rangle = e\mathcal{E}\langle n|z|1s\rangle$, so for which n is this nonzero? The operator z in spherical coordinates is $z = r\cos\theta$ which is proportional to the spherical harmonic $Y_1^0(\theta, \varphi)$. The selection rule is $\Delta l = \pm 1$. When z (which carries $l = 1$) acts on 1s (which carries $l = 0$) the only n states relevant are the ones with $l = 1$, i.e., $2p$ not $2s$. And, the selection rule for m is $\Delta m = 0$, which means only the $2p$ state with $m = 0$ is contributing. Consequently, what we need to evaluate is $\langle 210|z|100\rangle$:

$$\langle 210|z|100\rangle = \int \psi_{210}^{\dagger} z \psi_{100}\, d^3r = \int \frac{1}{4\sqrt{2\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta\, r \cos\theta \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} d^3r$$
$$\frac{1}{4\pi\sqrt{2}a_0^4} \int_0^{\infty} r^4 e^{-3r/2a_0} \int_0^{\pi} \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{1}{4\pi\sqrt{2}a_0^4} \frac{4!}{(\frac{3}{2a_0})^5} \frac{2}{3} 2\pi = \frac{256}{243\sqrt{2}} a_0 \tag{7}$$

Alas $e\mathcal{E}\langle n|z|1s\rangle = e\mathcal{E}\frac{256}{243\sqrt{2}}a_0$. The hydro energies are $E_1 = -\frac{me^4}{2\hbar^2}$ and $E_2 = -\frac{me^4}{8\hbar^2}$, derived from $E_n = -\frac{me^4}{2\hbar^2 n^2}$, hence, $E_1 - E_2 = \frac{3me^4}{8\hbar^2} = [a_0 = \hbar^2/me^2] = \frac{3e^2}{8a_0}$. The transition probability is then

$$P_{1s\to 2p(m=0)} = |\frac{\langle n|V|1s\rangle}{E_n^{(0)} - E_{1s}^{(0)}}|^2 = |\frac{e\mathcal{E}\frac{256}{243\sqrt{2}}a_0}{\frac{3e^2}{8a_0}}|^2 = \frac{2^{21}}{729^2}(\frac{\mathcal{E}a_0^2}{e})^2 \tag{8}$$

This problem was also asked in the entrance exam by Lev Gorkov's group (see problem 234). I do not know if the examiner, whether Landau or other, wants a discussion on the Stark effect here.